A second look at normal curvature

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When lecturing on normal curvature in a course on classical differential geometry of surfaces in Euclidean space, I was asked by a student for a geometric reason explaining why the principal directions are perpendicular to each other. I did not have an easy answer at hand. Moreover, it seemed easy to produce "counterexamples":

Consider a surface S defined as the graph of a smooth function $f:\mathbf{R}^2\to\mathbf{R}$. In polar coordinates, that function is supposed to be of the form $f(r,\theta)=r^2g(\theta)$ with $g:\mathbf{R}\to\mathbf{R}$ a smooth function with $g(t+\pi)=g(t)$ for all $t\in\mathbf{R}$, (a "radial parabolic" function).

The surface S has the XY-plane as its tangent plane at the origin O, since all curves $\big(r(t),\theta(t),r^2(t)g(\theta(t))\big)$ have horizontal tangents at O ($r=0$). The normal planes are thus all perpendicular to the XY-plane. A normal section, i.e., the intersection of the normal plane in direction θ with the surface S, consists therefore of the parabola with parameterization $\alpha_\theta(r)=r^2g(\theta)$ and normal curvature $k_n(\theta)=2g(\theta)$. Since we only assumed g to be smooth and to have period π, the Euler equations (1) relating normal curvatures to the principal curvatures seem to be violated in general. Nevertheless, the surfaces derived from our construction look quite "smooth", cf. Figure 1 and 2 – all figures are produced with the aid of Maple.

What is wrong? Well, the presentation in Euclidean coordinates of the function f defining the surface as its graph is $f\colon\mathbf{R}^2\to\mathbf{R}$,

$$f(x,y)=\begin{cases}0 & \text{if } (x,y)=(0,0),\\ (x^2+y^2)g\Big(\arctan\dfrac{y}{x}\Big) & \text{otherwise,}\end{cases}$$

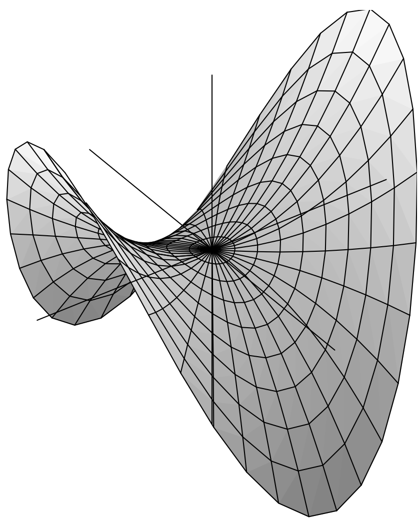

Figure 1: A smooth surface
$f(r, \theta) = r^2 \cos(2\theta)$

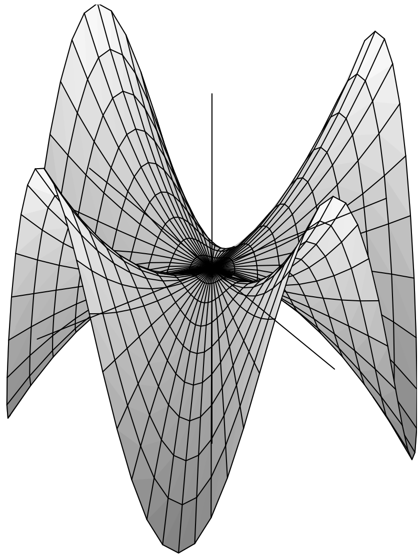

Figure 2: A nonsmooth surface
$f(r, \theta) = r^2 \cos(4\theta)$

and it is scarcely *smooth*. Smoothness, and in particular the fact that you are allowed to change the order under double differentiation, is essential in proving that the differential dN of the Gauss map N from the surface to the 2-sphere is *self-adjoint*. This property is crucial in the calculation of the normal curvatures and their relations to the principal curvatures.

In fact, we have the somewhat surprising result about such a function f:

Proposition 1. *A function* $f(x, y) = r^2 g(\theta)$ *is differentiable at* $(0, 0)$ if and only if *it is a quadratic form*

$$f(x, y) = Ax^2 + 2Bxy + Cy^2, \quad A, B, C \in \mathbf{R}.$$

Remark 2. A major part of the differential geometry of surfaces proceeds via the analysis of the best approximating quadratic forms at every point. Proposition 1 shows that the only radial parabolic functions that can be approximated by quadratic forms are the quadratic forms themselves. These *smooth* radial parabolic functions are thus particularly stiff since the coefficients $g(\theta)$ have to satisfy the Euler equations: They attain a minimal value $k_1 = g(\alpha)$ and a maximal value $k_2 = g(\alpha + \frac{\pi}{2})$ and

$$g(\alpha + \theta) = k_1 (\cos \theta)^2 + k_2 (\sin \theta)^2. \tag{1}$$

To prove the non-trivial part of the statement in Proposition 1 by elementary means, i.e., without using the Euler equations, we calculate the partial derivatives of f via polar coordinates:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{r} - \frac{\partial f}{\partial \theta}\frac{y}{r^2} = 2g(\theta)x - g'(\theta)y$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{y}{r} + \frac{\partial f}{\partial \theta}\frac{x}{r^2} = g'(\theta)x + 2g(\theta)y$$

for $(x,y) \neq (0,0)$ and $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. The second partial derivatives at $(0,0)$ are:

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \lim_{x\to 0}\frac{1}{x}\frac{\partial f}{\partial x}(x,0) = 2g(0)$$
$$\frac{\partial^2 f}{\partial y\,\partial x}(0,0) = \lim_{y\to 0}\frac{1}{y}\frac{\partial f}{\partial x}(0,y) = -g'\Big(\frac{\pi}{2}\Big)$$
$$\frac{\partial^2 f}{\partial x\,\partial y}(0,0) = \lim_{x\to 0}\frac{1}{x}\frac{\partial f}{\partial y}(x,0) = g'(0)$$
$$\frac{\partial^2 f}{\partial y^2}(0,0) = \lim_{y\to 0}\frac{1}{y}\frac{\partial f}{\partial y}(0,y) = 2g\Big(\frac{\pi}{2}\Big)$$

The smoothness of f has as a consequence that

$$g'(0) = \frac{\partial^2 f}{\partial x\,\partial y}(0,0) = \frac{\partial^2 f}{\partial y\,\partial x}(0,0) = -g'\Big(\frac{\pi}{2}\Big). \tag{2}$$

Moreover, the differentiability of $\partial f/\partial x$, resp. $\partial f/\partial y$ at $(0,0)$ is equivalent to the existence of the limits

$$\lim_{r\to 0}\frac{2\big(g(\theta)-g(0)\big)x - \big(g'(\theta)-g'(\pi/2)\big)y}{r} = \lim_{r\to 0}\big[2\big(g(\theta)-g(0)\big)\cos\theta - \big(g'(\theta)-g'(\pi/2)\big)\sin\theta\big]$$

and

$$\lim_{r\to 0}\frac{\big(g'(\theta)-g'(0)\big)x + 2\big(g(\theta)-g(\pi/2)\big)y}{r} = \lim_{r\to 0}\big[\big(g'(\theta)-g'(0)\big)\cos\theta + 2\big(g(\theta)-g(\pi/2)\big)\sin\theta\big].$$

These limits can only exist if the functions under the limit sign are constants, i.e., independent of θ, as well. Since they take the value 0 at $\theta = 0$, resp. at $\theta = \frac{\pi}{2}$, the function g has to satisfy the following two differential equations:

$$2\big(g(\theta)-g(0)\big)\cos\theta - \big(g'(\theta)-g'(\pi/2)\big)\sin\theta = 0 \tag{3}$$
$$\big(g'(\theta)-g'(0)\big)\cos\theta + 2\big(g(\theta)-g(\pi/2)\big)\sin\theta = 0. \tag{4}$$

Lemma 3. *A smooth function g with period π satisfies the differential equations (3) and (4) if and only if it is of the form*

$$g(\theta) = A(\cos\theta)^2 + 2B\cos\theta\sin\theta + C(\sin\theta)^2, A, B, C \in \mathbf{R}. \tag{5}$$

It is obvious that Lemma 3 yields Proposition 1.

Proof of Lemma 3: It is routine to check that any function g of the form (5) satisfies the differential equations (3) and (4).

To see the converse, we calculate $\frac{1}{2}[(3)\cos\theta + (4)\sin\theta]$ and obtain using (2):

$$g(\theta) = g(0)(\cos\theta)^2 + g'(0)\cos\theta\sin\theta + g(\pi/2)(\sin\theta)^2. \qquad \square$$

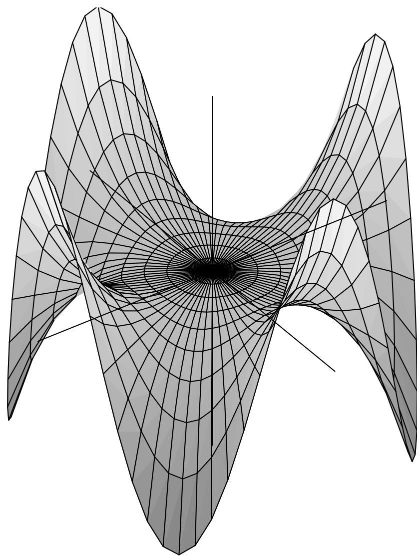

Figure 3: A surface with a flat point

Remark 4. There do exist less stiff functions of type $f(r,\theta) = h(r)g(\theta)$ whose graphs are smooth (twice differentiable) surfaces. But this happens at the expense of the normal curvatures of that surface at the origin being zero in every direction, i.e., the origin must be a flat point. An example is given by the function $f(r,\theta) = r^4\cos(4\theta)$, cf. Fig. 3 above.

Acknowledgment. I would like to thank the referee for pointing out a considerable shortcut in my original proof of Lemma 3.