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Integer Crossed Ladders; parametric representations and minimal integer values

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1 Introduction

The so-called ladder problems have attracted the interest of mathematicians through centuries. For an overview and a historical bibliography, see [3] and [4]. One such problem is the crossed ladders problem (CLP), which can be formulated as follows: Two ladders of length a and b lean against two vertical walls. The ladders cross each other at a point with distance c above the floor (see Figure 1). Determine the distance x between the walls and the height above the floor y and z where the ladders touch the walls.

The existence of integer solutions of this problem is well known (see [1], [6], [8] and [7]). However, the actual calculation of integer solutions using known methods is cumbersome. In this paper we introduce essentially simpler methods for calculating all possible integer solutions by means of four parameters. We also identify several interesting subclasses of CLP-solutions described by fewer parameters or obtained recursively by using sequences of Pell type. A study of minimal integer values is also included. Minimal integer values have been discussed earlier in [6] and [7]. In this paper we formulate new methods for identifying such values and present values smaller than the ones formerly published.

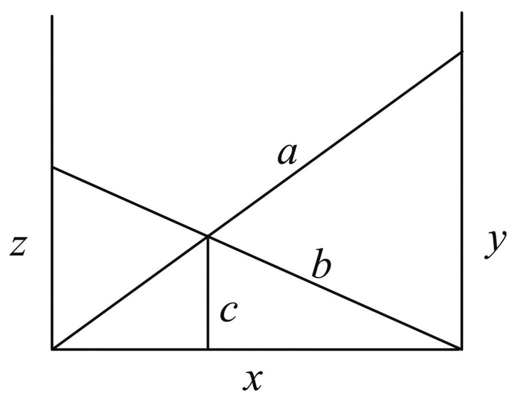

Figure 1: The ladder problem.

The paper is organized as follows. In Section 2 we give some remarks concerning Pythagorean triples and the trivial case $a = b$, $y = z$.Parametric representations of general solutions are collected in Section 3. An alternative method for parametric representation of several interesting subclasses can be found in Section 4. In Section 5 and Section 6 we study minimal integer values, in particular the minimal value of x, which appears in a certain subclass of CLP-solutions. The connection between this subclass and recursive formulae of type Pell numbers is found in Section 7.Similarly, in Section 8 we describe some subclasses of CLP-solutions containing minimum values for c and z.

2 On Pythagorean triples

By the above formulation of the crossed ladders problem we have that

$$\begin{cases} x^2 = a^2 - y^2 = b^2 - z^2, \\ c = \dfrac{yz}{y+z}. \end{cases} \tag{1}$$

The case $a = b$, $y = z$ is trivial, since the set of all Pythagorean triples (x, a, y) (i.e. triples of positive integers satisfying $x^2 = a^2 - y^2$) is precisely those triples of the forms

$$(x, a, y) = \left(m^2 - n^2, m^2 + n^2, 2mn\right), \tag{2}$$

$$(x, a, y) = \left(2mn, m^2 + n^2, m^2 - n^2\right), \tag{3}$$

where $m = M\sqrt{T}$, $n = N\sqrt{T}$, for some positive integers M, N and T. This fact is well known when n, m are integers and $\gcd(x, a, y) = 1$ (see [2]). The general case follows directly from this result by putting $T = \gcd(x, a, y)$. Since $c = y/2$, (2) will always give a solution to (1) and (3) will give a solution if $2 \mid m^2 - n^2$. Therefore, in the rest of this paper we will consider the non-trivial case assuming that

$$a > b \text{ (or } y > z). \tag{4}$$

Note also that if $x^2 = a^2 - y^2 = b^2 - z^2$, then it is clear that we obtain an integer solution to (1) by replacing (a, b, c, x, y, z) with $SG^{-1}(a, b, c, x, y, z)$, where $G = \gcd(a, b, c, x, y, z)$ and S is the scaling parameter:

$$S = \frac{y+z}{\gcd(yz, y+z)}.$$

We also observe that if

$$r = 2m_1n_1,\ s = m_1^2 - n_1^2,\ t = m_1^2 + n_1^2, \tag{5}$$

then

$$2r = m_2^2 - n_2^2,\ 2s = 2m_2n_2,\ 2t = m_2^2 + n_2^2,$$

where $m_2 = m_1 + n_1$, $n_2 = m_1 - n_1$. Moreover, replacing m_1 and n_1 with $\sqrt{2}m_1$ and $\sqrt{2}n_1$ in (5) we see that we also might put

$$2r = 2m_1n_1,\ 2s = m_1^2 - n_1^2,\ 2t = m_1^2 + n_1^2.$$

Remark 1. *This shows that for any Pythagorean triple (t, r, s), satisfying $t^2 = r^2 + s^2$, we have that $2(t, r, s)$ can be represented in both forms*

$$2(t,r,s) = \left(m^2+n^2, 2mn, m^2-n^2\right), \tag{6}$$

$$2(t,r,s) = \left(m^2+n^2, m^2-n^2, 2mn\right), \tag{7}$$

where $m = M\sqrt{T}$, $n = N\sqrt{T}$, for some positive integers M, N and T. As we will see in the following sections, this implies that we in principal may use only one of these representations as a tool for finding CLP-solutions.

Remark 2. *Letting m and n be integer values only, we are still able to represent any primitive Pythagorean triple (t, r, s) (satisfying $t^2 = r^2 + s^2$, $\gcd(r, s, t) = 1$), or its double $(2t, 2r, 2s)$ in both forms $(m^2 + n^2, 2mn, m^2 - n^2)$ and $(m^2 + n^2, m^2 - n^2, 2mn)$.*

3 Parametric representations of general solutions

Lemma 3. *Suppose that a, y, b, z, x and c are positive integers such that (1) holds. Then,*

$$(a, y, b, z, x) = Sw^{-1}(t_1s_2, r_1s_2, t_2s_1, r_2s_1, s_1s_2) \tag{8}$$

for some positive integers t_i, r_i, and s_i satisfying the conditions

$$\begin{cases} t_i^2 - r_i^2 = s_i^2, \\ w = \gcd(s_1, s_2), \\ S = \dfrac{r_1s_2 + r_2s_1}{\gcd(r_1s_2r_2s_1, r_1s_2 + r_2s_1)}. \end{cases} \tag{9}$$

Conversely, any set of integers of the form (8) satisfying (9) also satisfies (1).

Proof. Let $s_2' = \gcd(a, y)$ and $s_1' = \gcd(b, z)$. Let t_1, r_1, t_2 and r_2 be positive integers such that $a = t_1s_2'$, $y = r_1s_2'$, $b = t_2s_1'$ and $z = r_2s_1'$, where $\gcd(s_1', s_2') = 1$. By (1),

$$x^2 = \left(t_1^2 - r_1^2\right)s_2'^2 = \left(t_2^2 - r_2^2\right)s_1'^2.$$

Thus, we find that

$$t_i^2 - r_i^2 = s_i'^2w^2,\ i = 1, 2, \tag{10}$$

for some integer w. By putting $s_i = s_i'w$ we obtain (9). Conversely, if (9) holds, then it is seen directly by inspection that (1) holds. This completes the proof.

Since (t_i, r_i, s_i) of Lemma 3 is a Pythagorean triple, we know that $(t_i, r_i, s_i) = (t_i, u_i, v_i)$ or $(t_i, r_i, s_i) = (t_i, v_i, u_i)$, where

$$t_i = m_i^2 + n_i^2,\ u_i = 2m_i n_i,\ v_i = m_i^2 - n_i^2,$$

$$m_i = M_i\sqrt{S_i},\ n_i = N_i\sqrt{S_i},$$

for some positive integers M_i, S_i and N_i. Conversely, it is seen directly that (t_i, r_i, s_i) is a Pythagorean triple for any positive integers m_i, n_i. This leaves us with four alternative representations of the solution vector

$$(a, y, b, z, x) = Sw^{-1}(t_1 s_2, r_1 s_2, t_2 s_1, r_2 s_1, s_1 s_2), \tag{11}$$

from which we are able to reproduce all possible solutions of the crossed ladders problem in terms of the integer parameters m_1, m_2, n_1 and n_2. The four alternative representations are obtained by putting (t_i, r_i, s_i) equal to the following values:

1. $(t_1, r_1, s_1) = (t_1, u_1, v_1)$ and $(t_2, r_2, s_2) = (t_2, u_2, v_2)$,
2. $(t_1, r_1, s_1) = (t_1, u_1, v_1)$ and $(t_2, r_2, s_2) = (t_2, v_2, u_2)$,
3. $(t_1, r_1, s_1) = (t_1, v_1, u_1)$ and $(t_2, r_2, s_2) = (t_2, u_2, v_2)$,
4. $(t_1, r_1, s_1) = (t_1, v_1, u_1)$ and $(t_2, r_2, s_2) = (t_2, v_2, u_2)$.

For example, for the latter case (case 4) we obtain

$$a = Sm_2 n_2(m_1^2 + n_1^2), \tag{12}$$

$$b = Sm_1 n_1(m_2^2 + n_2^2), \tag{13}$$

$$c = \frac{m_1 n_1 m_2 n_2 (m_1^2 - n_1^2)(m_2^2 - n_2^2)}{\mathrm{gcd}_1}, \tag{14}$$

$$x = S2m_1 n_1 m_2 n_2, \tag{15}$$

$$y = Sm_2 n_2(m_1^2 - n_1^2), \tag{16}$$

$$z = Sm_1 n_1(m_2^2 - n_2^2), \tag{17}$$

where

$$S = \frac{m_2 n_2(m_1^2 - n_1^2) + m_1 n_1(m_2^2 - n_2^2)}{\mathrm{gcd}_1},$$

$$\mathrm{gcd}_1 = \mathrm{gcd}((m_1 n_1 m_2 n_2 (m_1^2 - n_1^2)(m_2^2 - n_2^2), m_2 n_2(m_1^2 - n_1^2) + m_1 n_1(m_2^2 - n_2^2)).$$

Note also that by (4) we may assume that $m_2 n_2(m_1^2 + n_1^2) > m_1 n_1(m_2^2 + n_2^2)$ (or equivalently that $m_1/n_1 > m_2/n_2$).

Remark 4. *Replacing* (t_i, r_i, s_i) *with* $(2t_i, 2r_i, 2s_i)$ *will change* (a, y, c, b, z, x) *obtained by (8) to* $(2a, 2y, 2c, 2b, 2z, 2x)$. *Hence, in view of Remark 1 the representation (12)-(17) will at least give us all even CLP-solutions. Thus, including the even solutions obtained from (12)-(17) and subsequently divided by 2, we obtain the complete set of CLP-solutions.*

4 Alternative representations

If

$$x^2 = a^2 - y^2 = b^2 - z^2, \tag{18}$$

then (x, a, y) and (x, b, z) can be represented as Pythagorean triples in four alternative ways, for example as

$$(x, a, y) = \left(p^2 - q^2, p^2 + q^2, 2pq\right), \tag{19}$$
$$(x, b, z) = \left(r^2 - s^2, r^2 + s^2, 2rs\right), \tag{20}$$
$$p^2 - q^2 = r^2 - s^2. \tag{21}$$

Hence, we obtain an integer solution to (1) by replacing (a, b, c, x, y, z), $c = yz/(y+z)$, with $S(a, b, c, x, y, z)$, where S is the scaling parameter:

$$S = \frac{pq + rs}{\gcd(pqrs, pq + rs)}.$$

This representation may be used to identify certain subclasses of solutions to (1). For example, observing that the identities

$$(2N_1^2 - N_2^2)^2 - \left(2N_1^2\right)^2 = \left(N_2^2\right)^2 - \left(2N_1N_2\right)^2, \tag{22}$$

$$(N_1^2 - 2N_2^2)^2 - \left(N_1^2\right)^2 = \left(2N_2\right)^2 - \left(2N_1N_2\right)^2, \tag{23}$$

are on the form (21), we obtain solutions by putting $(p, q, r, s) = (2N_1^2 - N_2^2, 2N_1^2, N_2^2, 2N_1N_2)$ or $(p, q, r, s) = (N_1^2 - 2N_2^2, N_1^2, 2N_2, 2N_1N_2)$.

Furthermore, by using the identities

$$(N_1^2 \pm N_2^2)^4 = N_1^8 \pm 4N_1^6N_2^2 + 6N_1^4N_2^4 \pm 4N_1^2N_2^6 + N_2^8,$$

$$(N_1^4 \pm N_2^4)^2 = N_1^8 \pm 2N_1^4N_2^4 + N_2^8,$$

$$(N_1^3N_2 \pm N_1N_2^3)^2 = N_1^6N_2^2 \pm 2N_1^4N_2^4 + N_1^2N_2^6,$$

we can obtain the following identities

$$4N_1^4N_2^4 = (N_1^4 + N_2^4)^2 - (N_1^4 - N_2^4)^2 = (N_1^3N_2 + N_1N_2^3)^2 - (N_1^3N_2 - N_1N_2^3)^2, \tag{24}$$

$$(N_1^4 - N_2^4)^2 = (N_1^2 + N_2^2)^4 - 4(N_1^3N_2 + N_1N_2^3)^2 = (N_1^4 + N_2^4)^2 - 4N_1^4N_2^4, \tag{25}$$

$$4(N_1^3N_2 - N_1N_2^3)^2 = 4(N_1^3N_2 + N_1N_2^3)^2 - 16N_1^4N_2^4 = (N_1^4 - N_2^4)^2 - (N_1^2 - N_2^2)^4. \tag{26}$$

By noting that each of the identities (24), (25) and (26) is on the form (18) we may obtain additional representations of subclasses of solutions for the crossed ladders problem. For example, by using (24) we see directly that

$$(x, a, y, b, z) = \left(2N_1^2N_2^2, N_1^4 + N_2^4, N_1^4 - N_2^4, N_1N_2(N_1^2 + N_2^2), N_1N_2(N_1^2 - N_2^2)\right)$$

is a solution to (18) for any integers N_1, N_2. Hence, replacing (a,b,c,x,y,z) with $S(a,b,c,x,y,z)$ (as before $c = yz/(y+z)$) where S is the scaling parameter

$$S = \frac{y+z}{\gcd(yz, y+z)} = N_1^2 + N_1N_2 + N_2^2,$$

we obtain one type of solution for the crossed ladders problem, namely the one of the form

$$(x,a,y,b,z) = S\left(2N_1^2N_2^2, N_1^4+N_2^4, N_1^4-N_2^4, N_1N_2(N_1^2+N_2^2), N_1N_2(N_1^2-N_2^2)\right), \tag{27}$$

$$c = N_1N_2(N_1^4 - N_2^4). \tag{28}$$

5 On minimum solutions

Minimum solutions has been considered earlier in [6] and [7]. However, the search for such quantities has mostly been based on tabulating solutions for increasing values of x. In this paper we will attack this problem from a different angle. The main idea is to represent sufficiently rich classes of solutions in such a way that minimum values can be identified more directly.

In [7] the following values are described as "the smallest possible solution": $x = 56$, $a = 119$, $b = 70$, $c = 30$, $y = 105$, $z = 42$. By our methods this solution is obtained by putting $m_1 = 4$, $n_1 = 1$, $m_2 = 2$, $n_2 = 1$ in the formulae (12-17), or $N_1 = 2$, $N_2 = 1$ in formulae (27) and (28).

It is not clear what is meant by "smallest possible solution" in the article [7]. By our investigation the above solution appears to have the property that the sum $x+a+b+c+y+z$ becomes smallest possible among all solutions. However, as we will show in this paper, none of the individual quantities x, a, b, c, y, z is minimal for this solution.

6 Minimum value for x

For the purpose of finding minimum values it is suitable to choose the parameters m_i and n_i such that scaling factor $S = 1$. In this way we avoid the calculation of S, which often is cumbersome. But more important, any subsequent scaling without knowing the value of S in advance, makes it hard to choose parameters such that the value of x gets minimal.

By (1) we have that $\gcd(yz, y+z) = y+z$. Thus, z and y can be represented as follows,

$$y = m(m+n), \ \ z = n(m+n), \tag{29}$$

where $m = p\sqrt{t}$ and $n = q\sqrt{t}$ for some integers p, q, t such that $p > q$. In order to verify this representation, we put $y = pk$ and $z = qk$, where $k = \gcd(y,z)$. Hence,

$$\gcd(p,q) = 1. \tag{30}$$

Letting $p+q=d\tau$, $pq=ds$, where $d=\gcd(p+q,pq)$, we find that $p=d_1s_1$, $q=d_2s_2$ for some integers d_1,d_2 and s_1,s_2, satisfying $d=d_1d_2$ and $s=s_1s_2$. Hence, $q=p+q-p=d_1d_2\tau-d_1s_1=d_1\left(d_2\tau-s_1\right)$. Thus, since $p=d_1s_1$ and $q=d_1\left(d_2\tau-s_1\right)$, (30) gives that $d_1=1$. Similarly, we have that $p=p+q-q=d_1d_2\tau-d_2s_2=d_2\left(d_1\tau-s_2\right)$, i.e. $p=d_2\left(d_1\tau-s_2\right)$ and $q=d_2s_2$, which, again according to (30), gives that $d_2=1$. Thus

$$\gcd(p+q,pq)=d=d_1d_2=1. \tag{31}$$

Moreover, since $\gcd(yz,y+z)=y+z$, we have that $yz=w(y+z)$ for some integer w, i.e. $pqk^2=wk(p+q)$. Thus, $pqk=w(p+q)$, and by (31), this gives that $k=t(p+q)$ for some integer t. Hence, $y=pk=tp(p+q)$ and $z=qk=tq(p+q)$. Putting $m=p\sqrt{t}$ and $n=q\sqrt{t}$ we therefore obtain (29).

A different way of showing (29) is by putting

$$y=c+r,\ z=c+s \tag{32}$$

for some integers r and s. From the identity $c=yz/(y+z)$ we obtain the simple relation

$$c^2=rs. \tag{33}$$

Next, we put $r=tr_1,\ \ s=ts_1$, where $t=\gcd(r,s)$ (such that $\gcd(r_1,s_1)=1$). By (33) we see that r_1s_1 is a perfect square, and since r_1 and s_1 do not have any prime factors in common, r_1 and s_1 have to be perfect squares, i.e. on the forms $r_1=p^2$, $s_1=q^2$. Thus

$$y=c+r=\sqrt{rs}+r=p\sqrt{t}(q\sqrt{t}+p\sqrt{t})=m(m+n), \tag{34}$$

$$z=c+s=\sqrt{rs}+s=q\sqrt{t}(p\sqrt{t}+q\sqrt{t})=n(m+n). \tag{35}$$

According to (19), (20), (21) and (29), one family of solutions to (1) is parametrized as follows:

$$x=2N_1N_2=2N_3N_4, \tag{36}$$

$$y=N_1^2-N_2^2=m(m+n), \tag{37}$$

$$z=N_3^2-N_4^2=n(m+n), \tag{38}$$

$$a=N_1^2+N_2^2,\ b=N_3^2+N_4^2,\ c=mn. \tag{39}$$

In addition, we restrict ourselves to the case

$$N_1+N_2=m+n,\quad N_1-N_2=m, \tag{40}$$

$$N_3=KM_3,\quad N_4=KM_4,\ K^2=n, \tag{41}$$

$$M_3^2-M_4^2=m+n, \tag{42}$$

$$n=2n_1. \tag{43}$$

Thus, we obtain that

$$m = 2M_3M_4 - n_1, \tag{44}$$

$$N_1 = m + n_1 = 2M_3M_4,\ N_2 = n_1, N_3 = M_3\sqrt{2n_1}, \tag{45}$$

$$N_4 = M_4\sqrt{2n_1}. \tag{46}$$

According to (42), (43) and (44) we have that

$$M_3^2 - M_4^2 - 2M_3M_4 - n_1 = 0 \tag{47}$$

i.e.

$$M_3 = M_4 + \sqrt{2M_4^2 + n_1}. \tag{48}$$

Now, choosing M_3, M_4 and n_1 such that (48) is satisfied, we directly obtain solutions of (1) without performing any subsequent scaling. However, for the purpose of finding minimal solutions, it may be necessary to divide the obtained solution (a, b, c, x, y, z) with $\gcd(a, b, c, x, y, z)$. By (36) and (45) we obtain $x = 4n_1M_3M_4$. For each fixed n_1, the smallest integers $M_3 > 0$, $M_4 > 0$ satisfying (48) are determined by the smallest integer $M_4 > 0$ making $2M_4^2 + n_1$ a perfect square. By inspection it is easy to see that x is smallest possible when $n_1 = 2$, $M_3 = 1$ and $M_4 = 3$. However, these values makes $y = z$, the trivial case, which contradicts our preliminary assumption (4). Except for this case, we observe that x is smallest possible when $n_1 = 1$, $M_3 = 2$ and $M_4 = 5$, giving the values

$$N_1 = 20, N_2 = 1, N_3 = 5\sqrt{2}; N_4 = 2\sqrt{2}, m = 19, n = 2,$$

which gives the solution

$$x = 40, a = 401, b = 58, c = 38, y = 399, z = 42.$$

This appears to be the solution which gives the smallest integer value of x among all possible solutions satisfying (4). In fact, by factorizing each integer between 1^2 and 39^2 and using that $x^2 = (a - y)(a + y) = (b - z)(b + z)$ it is possible to verify that there exists no integer solution for $x < 40$ such that $\gcd(yz, y + z) = y + z$.

7 Connection to Pell numbers

For $n_1 = 1$, we find an interesting connection between the class of solutions satisfying the condition (48) and the well-known *Pell numbers* $\{P_i\}_{i=0}^{\infty}$, which are defined recursively by

$$P_i = 2P_{i-1} + P_{i-2},\ P_0 = 0,\ P_1 = 1. \tag{49}$$

This sequence has the property that if $(x, y) = (P_{2i} + P_{2i-1}, P_{2i})$ then

$$x = \sqrt{2y^2 + 1}, \tag{50}$$

i.e. (x, y) is a solution to *Pell's Equation* $x^2 - 2y^2 = 1$. Concerning this and other basic properties of Pell numbers we refer to the literature (see e.g. [5] and the references given therein). Adding P_{2i} to (50) and using (49) we obtain that $P_{2i+1} = P_{2i} + \sqrt{2P_{2i}^2 + 1}$. Hence, comparing with (48) we obtain a class of solutions by putting

$$M_3 = P_{2i+1}, M_4 = P_{2i},\ m = 2P_{2i}P_{2i+1} - 1,\quad n = 2, \tag{51}$$

$$N_1 = 2P_{2i}P_{2i+1},\ N_2 = 1,\ N_3 = P_{2i+1}\sqrt{2},\ N_4 = P_{2i}\sqrt{2},$$

which gives the following integer solution to our problem:

$$a = 4P_{2i}^2 P_{2i+1}^2 + 1,\ b = 2(P_{2i+1}^2 + P_{2i}^2),\ c = 2(2P_{2i}P_{2i+1} - 1), \tag{52}$$

$$x = 4P_{2i}P_{2i+1},\ y = 4P_{2i}^2 P_{2i+1}^2 - 1,\ z = 2(P_{2i+1}^2 - P_{2i}^2). \tag{53}$$

For the calculation of the Pell numbers, the following explicit formula may be useful

$$P_i = \frac{(1+\sqrt{2})^i - (1-\sqrt{2})^i}{2\sqrt{2}}.$$

We also observe that $b+z$, $b-z$ and b^2-z^2 are perfect squares equal to $(2P_{2i+1})^2$, $(2P_{2i})^2$ and $(4P_{2i}P_{2i+1})^2$, respectively.

By choosing different values of n_1 we obtain other families of integer solutions of (1) generated by pairs (M_3, M_4) of the type $M_3 = P_k$, $M_4 = P_l$. For example, if $n_1 = 2$ we obtain the same definition of the sequence P_i as above except that $P_0 = 1$, i.e.

$$P_i = 2P_{i-1} + P_{i-2},\ P_0 = 1,\ P_1 = 1, \tag{54}$$

with corresponding explicit formula

$$P_i = \frac{(1+\sqrt{2})^i + (1-\sqrt{2})^i}{2}.$$

In this case we put $M_3 = P_{2i}$, $M_4 = P_{2i-1}$. All obtained solutions for $n_1 = 2$ may be divided by 8.

The minimal non-trivial solution in the subclass generated from the case $n_1 = 2$ is

$$x = 119, a = 7081, b = 169, c = 118, y = 7080, z = 120.$$

As a curiosity we want to add that this appears to be the smallest possible solution satisfying $a - y = 1$. In addition, this also appears to be the smallest possible solution where three elements have consecutive values like $c = 118, x = 119, z = 120$. It is possible to show that all solutions generated from the case $n_1 = 2$ (after division by the greatest common divisor) have the property $a - y = 1$, $z - x = 1$ and c, x, z have consecutive values. As for the case $n_1 = 1$, $b + z$, $b - z$ and $b^2 - z^2$ are perfect squares, equal to $(P_{2i})^2, (P_{2i-1})^2$ and $(P_{2i}P_{2i+1})^2$, respectively.

It can be shown that there exists no integers M_3 and M_4 satisfying (48) for $n_1 = 5$. For $n_1 = 7$ we have that $M_3 = P_{2i+1}$, $M_4 = P_{2i}$, where $\{P_i\}$ are the following Pell type of numbers,

$$P_{2i+1} = 2P_{2i} + P_{2i-3}, P_{2i} = 2P_{2i-3} + P_{2i-4}, P_0 = 0, P_1 = 1.$$

For $n_1 = u^2$, where u is an integer, $M_3 = P_{2i+1}, M_4 = P_{2i}$, where $\{P_i\}$ is given by

$$P_i = 2P_{i-1} + P_{i-2}, P_0 = 0, P_1 = u.$$

8 Minimum values for c and z

It is known that 14 and 15 are the minimum values for c and z, respectively. Similarly as above we will in this section develop a representation of a sufficiently rich class of solutions, from which these minimal solutions are derived more directly than the cumbersome tabulation approach.

It is reasonable to assume that the minimum values for c and z appear when the differences $z - c$ and $b - x$ are small (see Figure 1). Recall that by putting

$$y - c = r, \tag{55}$$

$$z - c = s,$$

we obtain from the identity $c = yz/(y + z)$ the simple relation

$$c^2 = rs. \tag{56}$$

We restrict ourselves to the case

$$z - c = s = 1, \quad b - x = 1. \tag{57}$$

Let $x = x_1 + x_2$, where x_1 denotes the horizontal distance between the left wall and the intersection point. By comparing equilateral triangles we obtain the relations $x_1/(z - c) = x_1 = x_2/c$ (which implies that $x = x_1(1 + c)$), and $b/x = \sqrt{(z-c)^2 + x_1^2}/x_1$, i.e.

$$\frac{x+1}{x} = \frac{x_1(1+c)+1}{x_1(1+c)} = \frac{\sqrt{1+x_1^2}}{x_1}. \tag{58}$$

Solving (58) we find that

$$x = \frac{1}{2}c\,(2 + c)\,. \tag{59}$$

By (55), (56), (57) and (59) we have that

$$y = c^2 + c, \quad z = 1 + c, \;\; b = \frac{1}{2}c\,(2 + c) + 1, \tag{60}$$

and, hence, by the formula $a^2 = x^2 + y^2$, we find that

$$a^2 = \frac{5}{4}c^4 + 3c^3 + 2c^2,$$

which shows that c is even, i.e. of the form $c = 2p$. Accordingly,

$$a^2 = 4p^2(5p^2 + 6p + 2), \tag{61}$$

$$b = 2p^2 + 2p + 1,\ \ c = 2p, \tag{62}$$

$$x = 2p(p+1), \quad y = 2p(2p+1),\ \ z = 2p + 1. \tag{63}$$

Due to (61) we have that $5p^2 + 6p + 2$ is a perfect square. Besides for $p = -1$ (which we ignore since it represents the trivial case), the smallest positive value of $c = 2p$ satisfying this condition is obtained for $p = 7$. This value gives us the following integer solution

$$a = 238, b = 113, c = 14, x = 112, y = 210, z = 15, s = 1, r = 196,$$

for which both z and c are minimal. We also note that this solution also gives the minimum values for $b - x$ and $z - c$.

A sequence of integer values of p satisfying the above condition can be found by putting

$$5p^2 + 6p + 2 = (2p + q)^2 \tag{64}$$

for some integer q, which yields the following explicit expressions

$$p = 2q - 3 \pm \sqrt{5q^2 - 12q + 7}, \tag{65}$$

$$q = -2p \pm \sqrt{5p^2 + 6p + 2}. \tag{66}$$

We note that if one of the values given by (65) is an integer (for a fixed integer value of q), so is the other one. The same holds for (66). This explains why we choose to express the right side of (64)) on the form $(2p + q)^2$. We may now produce an infinite family of pairs $(p, q) = (p_{2i+1}, q_{2i})$ satisfying the identity (64) by using the recursive formulae

$$p_{2i+1} = 2q_{2i} - 3 + \sqrt{5q_{2i}^2 - 12q_{2i} + 7}, \quad q_0 = 1, \tag{67}$$

$$q_{2i} = -2p_{2i-1} + \sqrt{5p_{2i-1}^2 + 6p_{2i-1} + 2}. \tag{68}$$

From these formulae we obtain the following values:

$$q_0 = 1 \to p_1 = -1 \to q_2 = 3 \to p_3 = 7 \to$$

$$q_4 = -31 \to p_5 = -137 \to q_6 = 579 \to p_7 = 2447 \to \ldots \quad .$$

Note that by inserting any value $p = p_i$, obtained in this way, into (61), (62) and (63) we obtain solutions of our problem satisfying the condition (57). Interestingly, all these solutions have the property that

$$y/z = c, \tag{69}$$

which follows directly by (60).

We also want to add that the above sequences also can be obtained by the following recursive formulae

$$q_{2i} = -4p_{2i-1} - q_{2i-2}, \tag{70}$$

$$p_{2i+1} = 4q_{2i} - p_{2i-1} - 6, \tag{71}$$

$$q_0 = 1,\ p_1 = -1. \tag{72}$$

When the condition (57) is satisfied we obtain from (69) that $F \equiv y/z$ is an integer. For the class corresponding to $n_1 = 1$ in Section 7, we obtain from (37), (38) and (51) that

$$F = \frac{y}{z} = \frac{m}{n} = \frac{2P_{2i}P_{2i+1} - 1}{2}, \tag{73}$$

which never is an integer. On the other hand, for the class corresponding to $n_1 = 2$,

$$F = \frac{y}{z} = \frac{m}{n} = \frac{2P_{2i}P_{2i-1} - 2}{4} = \frac{P_{2i}P_{2i-1} - 1}{2}. \tag{74}$$

By induction it is directly seen from (54) that the sequence $\{P_i\}$ consists of odd numbers. Thus, $F = y/z$ is an integer according to (74).

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