Real Toric Surfaces

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1. INTRODUCTION

Almost every course in topology starts with the notion of the manifold. The variety of examples used to visualize this important concept includes the projective space. This simple non-trivial example is particularly useful, because it allows to write down explicitly formulas for all related notions such as charts, trivializations, and transition functions. It is also important that these formulas are elementary, they use only monomial functions. This property identifies an important class of algebraic manifolds called *toric varieties*. Apart from affine, projective, and weighted projective spaces, they include their products, Hirzebruch surfaces and many other interesting manifolds.

Another kind of examples used for the demonstration of basic properties of manifolds is the so-called *classical surfaces*: the sphere, the torus, the Klein bottle, the double torus, and so on. In general, a classical surface is a sphere, or a connected sum of either tori or real projective planes. The process of triangulating or gluing them from their fundamental polygons can be easily demonstrated to the audience, which further enhances their educational value.

A question arises here whether we can analyze a real two-dimensional manifold from these both angles. In other words, if there is a compact smooth real surface which admits an atlas with monomial transition mappings, i.e., whether an example of the real toric surface can be found.

The obvious answer is that such a surface exists and that it is the torus. The latter is the real part of the product of two Riemann spheres. Since this product is a toric variety, the torus is a real toric variety. However, this is the only example of a smooth *orientable* compact surface admitting such an atlas. In this note we offer an elementary proof of this fact.

2. CONSTRUCTION OF TORIC VARIETY

Definition. A complex toric variety is an irreducible variety X such that

1. The algebraic torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$ is a Zariski open subset of X;
2. The multiplicative action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

As already noted, the most basic examples of toric varieties are the algebraic torus $(\mathbb{C}^*)^n$, the affine space $\mathbb{C}^n$, and the projective space $\mathbb{P}_n$. In general, a toric variety

X consists of $(\mathbb{C}^*)^n$ plus a finite number of toric hypersurfaces, which are also toric varieties. This multilevel structure is reflected in a combinatorial object called *fan*.

A fan Σ is defined as a finite collection of cones in $\mathbb{R}^n$ with the following properties:

- each cone $\sigma \in \Sigma$ is a strongly convex rational (with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$) polyhedral cone;
- all faces of $\sigma \in \Sigma$ belong to Σ;
- if faces σ, $\tau \in \Sigma$, then $\sigma \cap \tau$ is a face of both.

Every cone σ from Σ defines an n-dimensional affine toric variety U_σ, and the toric variety X_Σ is obtained by gluing together U_σ and U_τ along $U_{\sigma\cap\tau}$ for all σ, $\tau \in \Sigma$. It should be stressed here that a fan completely determines a toric variety, therefore, the question of varieties can be translated into the question of fans and solved by elementary means.

In this note we choose a different construction of toric varieties from the one mentioned above. This approach was developed independently by several people (see [1]). It generalizes the quotient representation of the projective space as a set of lines passing through the origin in an affine space. A toric variety in this case is a set of certain monomial surfaces in an affine space passing through a special collection of coordinate planes. For later use, we outline below the main points of this construction.

Let $v_1, \dots, v_d \in \mathbb{Z}^n$ be the primitive elements (minimal integer generators) of one-dimensional cones of the fan Σ in $\mathbb{R}^n$. Assign a variable t_i to each vector v_i. In the space $\mathbb{C}^d$ of variables $t = (t_1, \dots, t_d)$ we consider the variety

$$Z(\Sigma) = \big\{ t \in \mathbb{C}^d : \prod_{v_i \notin \sigma} t_i = 0 \quad \forall\, \sigma \in \Sigma \big\}.$$

In effect, to define this variety we may restrict ourselves to monomials associated with maximal cones of Σ. The function of this union of coordinate planes is parallel to that of the origin in the case of the projective space.

The monomial surfaces in $\mathbb{C}^d$ are the orbits of the multiplicative action of the group G on $\mathbb{C}^d \setminus Z(\Sigma)$. The group G is a subgroup of $(\mathbb{C}^*)^d$ and is defined by

$$G = \left\{ g = (\mu_1, \dots, \mu_d) \in \mathbb{C}^d \colon \prod_{i=1}^{d} \mu_i^{\langle m,\, v_i \rangle} = 1 \quad \forall m \in \mathbb{Z}^n \right\},$$

although it suffices to take only basis elements of the lattice for m. Thus, we have only n relations, and G is isomorphic to $(\mathbb{C}^*)^{d-n}$.

The complex toric variety is defined as a quotient $(\mathbb{C}^d \setminus Z(\Sigma))/G$, but this quotient is a smooth manifold if and only if the fan Σ is simplicial and primitive. In other words, if each k-dimensional cone is generated by $k+1$ one-dimensional cones and the list of their primitive generators may be completed to a basis of $\mathbb{Z}^n$. Furthermore, X_Σ is compact if the supports of all cones of Σ cover the whole space $\mathbb{R}^n$. We refer to the tutorial [2] for the details of both constructions and numerous examples.

Either construction can be transposed to the field of real numbers $\mathbb{R}$ producing a *real* toric variety, which is a real part of the corresponding complex one.

We reformulate now our question in terms of fans. We are looking for a complete simplicial primitive fan Σ in $\mathbb{R}^2$ such that the corresponding real toric variety is an orientable surface. Order the generators of one-dimensional cones of Σ counterclockwise. A two-dimensional cone generated by a pair of adjacent vectors v_i, v_j is primitive if and only if the determinant

$$\Delta_{ij} := v_{i1}v_{j2} - v_{i2}v_{j1} = 1. \tag{1}$$

It should be pointed out that a fan in $\mathbb{R}^2$ with one-dimensional generators $v_1, \ldots, v_d$ satisfying (1) is already complete and simplicial. Moreover, it can be also shown that the resulting toric variety is orientable if and only if

$$\text{for every generator } v_i = (v_{i1}, v_{i2}) \text{ the number } |v_i| = v_{i1} + v_{i2} \text{ is odd.} \tag{2}$$

A justification of this fact may be found in [3], where the real toric varieties were successfully utilized.

Condition (2) places a severe restriction on the structure of fans in $\mathbb{R}^2$ and plays a crucial role in the next section. At the same time, without this assumption our question presents a difficult problem, which is not yet solved.

3. THE FAN OF THE ORIENTABLE TORIC SURFACE

Let us start by introducing the following integer vectors

$$e_1 = (1,0),\ e_2 = (0,1),\ e_3 = (-1,0),\ e_4 = (0,-1),$$
$$e_5 = (1,1),\ e_6 = (-1,1),\ e_7 = (-1,-1),\ e_8 = (1,-1),$$

and open half-planes

$$\Pi_k = \{v \in \mathbb{R}^2 : \langle v, e_k \rangle > 0\}, \quad k = 1, \ldots, 8.$$

Lemma 1. *Let Σ be a fan in $\mathbb{R}^2$ satisfying* (1) *and* (2)*. Suppose that e_k lies in the interior of the cone $\sigma = \sigma(v_i, v_j)$ generated by v_i and v_j. Then either $\sigma \not\subset \Pi_k$, or σ is a coordinate quadrant.*

Proof. Consider first the case when $k = 1, \ldots, 4$. The vector e_k lies on an axis, from which necessarily follows that $\sigma \not\subset \Pi_k$. Otherwise, the generators v_i and v_j of σ would lie in adjacent quadrants, hence the products $v_{i1}v_{j2}$ and $v_{i2}v_{j1}$ differ from zero and have different signs. This implies that the absolute value $|\Delta_{ij}| > 1$, which contradicts (1).

Examine now the case when vector e_k lies on the bisector of a quadrant, i.e., $k = 5, \ldots, 8$. Obviously, if σ is a quadrant containing e_k, it satisfies conditions (1), (2) and lies entirely in Π_k.

Suppose now that σ is not a quadrant and $\sigma \subset \Pi_k$. Then there are three possible ways to arrange the generators v_i and v_j and the coordinate quadrant I_k containing e_k:

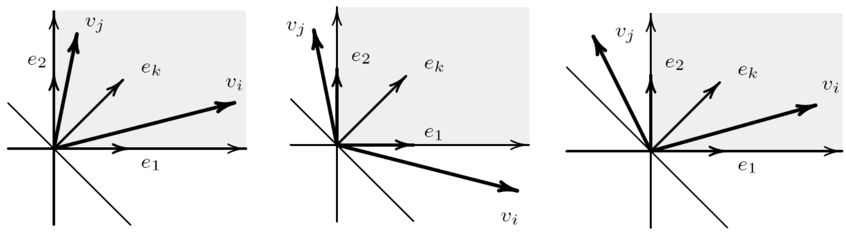

Figure 1.

a) the generators v_i and v_j lie in I_k;

b) the quadrant I_k lies entirely in σ;

c) one of the generators of σ lies inside I_k, while the other does not.

It is easily seen that the last case has already been ruled out, since the cone σ contains the vector e_l, $1 \leq l \leq 4$ and lies entirely in the corresponding open half-plane Π_l.

For the cases a) and b), notice that

$$\left|\Delta_{ij}\right| = \left|v_{i1}v_{j2} - v_{i2}v_{j1}\right| = \left||v_{i1}v_{j2}| - |v_{i2}v_{j1}|\right|.$$

This follows from the fact that the products $v_{i1}v_{j2}$ and $v_{i2}v_{j1}$ have the same sign, because the generators lie either in the same quadrant or in quadrants symmetrical about the origin. More precisely, in these cases one of the generators of $\sigma(v_i, v_j)$ belongs to the set $D_1 = \{(x,y) : |x| > |y|\}$, whereas the other to the set $D_2 = \{(x,y) : |x| < |y|\}$. Even more specifically, let v_i be in D_1, then we may write

$$|v_{i1}| = p + |v_{i2}|,$$
$$|v_{j2}| = q + |v_{j1}| \text{ with positive integers } p \text{ and } q.$$

Substituting these equalities in the expression for $\left|\Delta_{ij}\right|$ we arrive at

$$|\Delta_{ij}| = \Big|pq + |v_{i2}|q + |v_{j1}|p\Big| = pq + |v_{i2}|q + |v_{j1}|p.$$

It should be taken into account that σ is not a quadrant, that is, v_{i2} and v_{j1} are non-zero. For that reason $|\Delta_{ij}| > 1$, which contradicts (1). This concludes the proof. □

Lemma 2. *A cone σ satisfying* (1) *and* (2)*, and generated by $v_i = (v_{i1}, -1)$ and $v_j = (v_{j1}, -1)$ cannot be decomposed into a fan with the same properties.*

Proof. As the coordinates v_{i1} and v_{j1} are necessarily even, we can write $v_{i1} = 2p$, $v_{j1} = 2q$, $p, q \in \mathbb{Z}$.

Assume that there exists a refinement of σ, and every cone of it satisfies (1) and (2). Consider the vector $(2p+1, -1)$. It lies in σ and cannot be a generator,

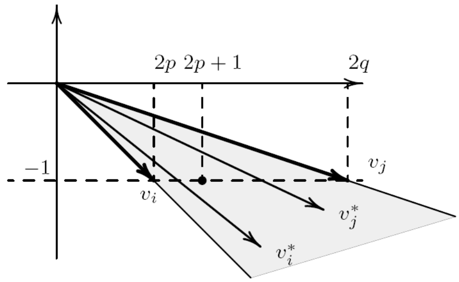

Figure 2.

since the sum of its coordinates is even. Therefore, it lies in some cone $\sigma^*(v_i^*, v_j^*)$ of the decomposition (see Figure 2), from which follows that $(2p+1, -1)$ is a linear combination of the generators v_i^* and v_j^* with positive coefficients, that is to say, there exist positive integers α and β such that

$$\begin{cases} \alpha v_{i1}^* + \beta v_{j1}^* = 2p+1, \\ \alpha v_{i2}^* + \beta v_{j2}^* = -1. \end{cases}$$

On the other hand, $v_{i2}^* \leq v_{i2} = -1$ and $v_{j2}^* \leq v_{j2} = -1$, so that the second equality cannot hold. The consequence is that σ cannot be further decomposed. □

As a next step we will impose one additional restriction on fans. The reason for placing this restriction is that two fans encode the same variety if there is a unimodular linear transformation of $\mathbb{Z}^2$ turning one fan in $\mathbb{R}^2$ into another. There is a simple explanation for this: such a transformation of fans induces a monomial isomorphism of the corresponding toric varieties. In order to avoid this ambiguity, we shall consider fans only in their *canonical form*, i.e., fans containing the positive quadrant as a two-dimensional cone.

This restriction does not eliminate any possible variety we are looking for, as demonstrated by the following lemma.

Lemma 3. *Let Σ be a fan satisfying* (1) *and* (2) *and $\sigma(v_i, v_j)$ be any of its cones. Then there exists a unimodular linear transformation turning σ into the positive quadrant and preserving properties* (1) *and* (2) *of the fan.*

Proof. It is obvious that the transformation of $\mathbb{R}^2$ with the matrix

$$A = \begin{pmatrix} v_{j2} & -v_{j1} \\ -v_{i2} & v_{i1} \end{pmatrix}$$

is unimodular and turns σ into the positive quadrant. Property (1) of the transformed fan follows immediately from the properties of determinants, and property (2) is equally easy to prove. □

At this stage, we are ready to prove the following key proposition that establishes the structure of the fan satisfying (1) and (2).

Proposition 1. *The canonical form of a complete fan* Σ *in* $\mathbb{R}^2$ *satisfying* (1) *and* (2) *has only four one-dimensional cones generated by* e_1, e_2, e_3, *and* $v = (a, -1)$, *or by* e_1, e_2, e_4, *and* $v = (-1, a)$ *with an even* a (*see Figure 3*).

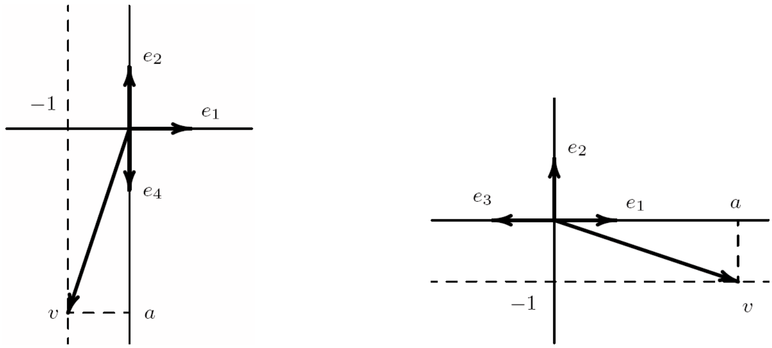

Figure 3.

Proof. Assume that there exists a fan with prescribed properties, and its one-dimensional generators are numbered counterclockwise, starting from e_1. The vector e_1 becomes v_1, e_2 becomes v_2, and the last one, preceding e_1, turns into v_d.

As the fan Σ is complete, it has a cone $\sigma(v_i, v_j)$ containing e_8. According to Lemma 1, the following three arrangements of vectors v_i, v_j, and e_k are possible (see Figure 4):

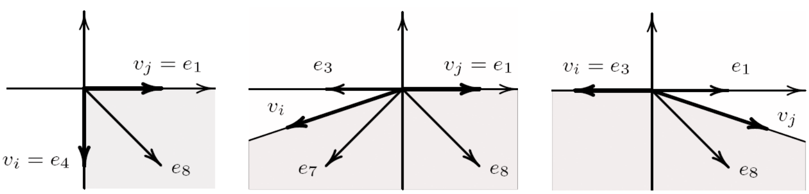

Figure 4.

a) the cone $\sigma(v_i, v_j)$ coincides with the quadrant generated by e_4 and e_1;

b) the vector v_j coincides with e_1, and v_i lies inside the cone $\sigma(e_3, e_7)$;

c) the vector v_i coincides with e_3, and v_j lies inside the cone $\sigma(e_8, e_1)$.

In the first case, the fan Σ already has two two-dimensional cones $\sigma(e_1, e_2)$ and $\sigma(e_1, v_d)$ with the generator v_d coinciding with e_4. Condition (1) imposed on these cones yields $v_3 = (-1, a)$ and $v_{d-1} = (-1, b)$, but the cone generated by these two vectors does not possess this property. Moreover, according to Lemma 2, this cone cannot be further decomposed. Thus, the half-plane Π_3 contains only one generator $v_3 = (-1, a)$, and cones of Σ are generated by $e_1, e_2, e_4, (-1, a)$.

In the second case, the fan Σ contains two cones $\sigma(e_1, e_2)$ and $\sigma(v_d, e_1)$ with $v_d = v_i$. Consider a refinement of the cone $\sigma(e_2, v_d)$. The vector v_{d-1} cannot lie in the interior of the quadrant $\sigma(e_2, e_3)$, since this possibility is excluded by Lemma 1, and Lemma 2 states that v_{d-1} cannot lie inside of $\sigma(e_6, e_3)$. The only possibility left is that v_{d-1} is e_3. The next step is to subdivide the quadrant $\sigma(e_2, e_3)$ but, as we have seen, this contradicts Lemma 2. As a result, the fan Σ has only four generators: e_1, e_2, e_3, and $(a, -1)$.

In the third case, we are already in a position to subdivide $\sigma(e_2, e_3)$, which contradicts Lemma 2. □

4. CONSTRUCTING A VARIETY FROM THE FAN

In this section, we show that the fans obtained in Proposition 1 always encode, up to a homeomorphism, the torus.

Proposition 2. *Any smooth compact orientable real toric surface is homeomorphic to the torus.*

Proof. First note that toric surfaces corresponding to the fans with the generators e_1, e_2, e_3, $v = (a, -1)$ and e_1, e_2, e_4, $v = (-1, a)$ are isomorphic since they differ only by rotation. We shall choose the first set of generators. The corresponding toric variety is defined as a quotient of $\mathbb{R}^4 \setminus Z$ by the action of the group G. The set Z is the union of the two planes $\{t_1 = t_3 = 0\}$ and $\{t_2 = t_4 = 0\}$ in $\mathbb{R}^4$ and $g = (\mu_1, \mu_2, \mu_3, \mu_4) \in G$ if $\mu_1/\mu_3\mu_4^{-a} = \mu_2/\mu_4 = 1$. Denoting $\lambda_1 = \mu_3$ and $\lambda_2 = \mu_4$, we may write the action of g on $t = (t_1, t_2, t_3, t_4)$ as

$$g \cdot t = (\lambda_1\lambda_2^{-a}t_1, \lambda_2 t_2, \lambda_1 t_3, \lambda_2 t_4), \quad (\lambda_1, \lambda_2) \in (\mathbb{R} \setminus \{0\})^2. \tag{3}$$

We claim that the Cartesian product of two semicircles in $\mathbb{R}^4$

$$E = \{t_1^2 + t_3^2 = 1, t_2^2 + t_4^2 = 1, t_3 \geq 0, t_4 \geq 0\}$$

is a screen for the action of G, or equivalently, that every orbit of the group action intersects this set.

Indeed, orbit (3) of t intersects E if the following system of equations and inequalities in λ has a real solution:

$$\begin{cases} (\lambda_1\lambda_2^{-a}t_1)^2 + (\lambda_1 t_3)^2 = 1, & \lambda_1 t_3 \geq 0, \\ (\lambda_2 t_2)^2 + (\lambda_2 t_4)^2 = 1, & \lambda_2 t_4 \geq 0. \end{cases} \tag{4}$$

One can easily see that

$$\lambda_1 = \pm\sqrt{\frac{1}{\lambda_2^{-2a}t_1^2 + t_3^2}}, \quad \lambda_2 = \pm\sqrt{\frac{1}{t_2^2 + t_4^2}} \tag{5}$$

solves the equations. Since (t_1, t_3) and (t_2, t_4) are from $\mathbb{R}^2 \setminus \{0\}$, these solutions always exist. Besides, the inequalities in (4) determine the choice of a sign of the roots if $t_3 \neq 0$, $t_4 \neq 0$. This proves that if an orbit intersects the screen E in an interior point then this point is unique.

We shall now turn to the case when an orbit meets E at a boundary point. Fix a point $P = (t_1, t_2, t_3, t_4) \in \partial E$. The coordinates of P satisfy $t_1^2 + t_3^2 = 1$, $t_2^2 + t_4^2 = 1$, consequently, (5) implies that the orbit of P intersects the screen E if $\lambda_1 = \pm 1$, $\lambda_2 = \pm 1$, that is, in no more than four points, including P (for $\lambda_1 = \lambda_2 = 1$).

Since $P \in \partial E$, at least one of the coordinates t_3, t_4 is equal to zero. Consider two cases:

a) Assume that $t_3 = 0,\ t_4 \neq 0$, then $P = (\pm 1, t_2, 0, t_4)$. The inequality $\lambda_2 t_4 \geq 0$ implies that system (4) has only two solutions: $\lambda_1 = \pm 1$, $\lambda_2 = 1$. From this we conclude that the orbit of P also intersects the screen E in another point

$$P' = (\lambda_1 \lambda_2^{-a} \cdot (\pm 1), \lambda_2 t_2, \lambda_1 \cdot 0, \lambda_2 t_4) = (\mp 1, t_2, 0, t_4).$$

(see Figure 5). These two points must be identified.

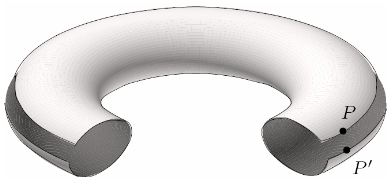

Figure 5.

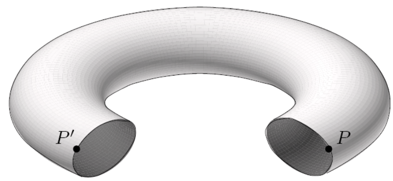

Figure 6.

b) Assume now that $t_3 \neq 0,\ t_4 = 0$. Analogously, the solutions to system (4) are $\lambda_1 = 1,\ \lambda_2 = \pm 1$, therefore, the point E must be identified with the point

$$P' = (\lambda_1 \lambda_2^{-a} t_1, \lambda_2 \cdot (\pm 1), \lambda_1 t_3, \lambda_2 \cdot 0) = (t_1,\ \mp 1,\ t_3,\ 0),$$

(see Figure 6).

Thus, in the screen, which is homeomorphic to a square, we identify the opposite sides in such a way that the resulting surface is the torus (see Figure 7).

We may add here that if a were odd, in case b) the sides would be glued differently, resulting in the Klein bottle, a non-orientable surface (see Figure 8). □

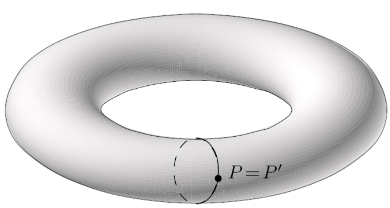

Figure 7.

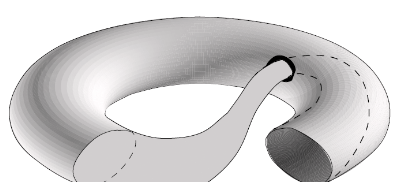

Figure 8.

REMARK. This results follow immediately from Proposition 1 as soon as we notice that those fans encode complex toric varieties called Hirzebruch surfaces. Topologically every Hirzebruch surface is a fiber product with the base $\mathbb{CP}_1$ and the fiber $\mathbb{CP}_1$. The real part of it is either the torus or the Klein bottle.

References

[1] D. Cox *The homogeneous coordinate ring of a toric variety*, J. Alg. Geometry **4** (1995), 17–50.

[2] D. Cox *What is a toric variety?* Contemporary Math., vol. 334, 203–224.

[3] T.O. Ermolaeva, A.K. Tsikh *Integration of rational functions over* $\mathbb{R}^n$ *by means of toric compactifications and higher-dimensional residues*, Sb. Math. **187** (1996), №9, 1301–1318.