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Alternative route: from van Schooten to Ptolemy

Andrew Percy and D. G. Rogers

SASE, Monash University
Northways Road, CHURCHILL
AUSTRALIA, Vic 3842
andrew.percy@sci.monash.edu.au

1 Van Schooten's Theorem (1646)

Frans van Schooten (1615–1660) was influential in the mathematical circle of his time in several ways, but not least in the selection or formulation of choice results. In this regard, van Schooten acted as a significant conduit for the ideas of contemporaries, notably René Descartes (1596–1650), but also Pierre de Fermat (1601–1665), to reach a wider audience — Isaac Newton (1642–1727), for one, read van Schooten's work with close attention in the mid-1660s, drawing on it in his Lucasian lectures at Cambridge in the decade from 1673 (compare [22, esp. (a), n. 7, p. x]; another Dutch source for Newton was the algebra textbook of 1661 by Gerard Kinckhuysen (1625–1666)).

One of van Schooten's theorems, in his *De Organica Conicarum Sectionum In Plano Descriptione, Tractatus* [19], of 1646, brings to light a curious property of equilateral triangles (see also [14, (b)] and [11]). Let $\triangle ABC$ be an equilateral triangle. If P is a point on the arc AB of the circumcircle of $\triangle ABC$ opposite C, as in Figure 1(i), then van Schooten shows that

$$CP = AP + BP. \tag{1}$$

The fact that $APBC$ is a cyclic quadrilateral is something with which to set to work. We might recall a staple in school geometry textbooks in years gone by (see, for example, [12, 7, 8]; and compare [6]), the theorem of Ptolemy (c. 85–c. 165) that once served as the foundation of trigonometric computation: the rectangle contained by the diagonals of a cyclic quadrilateral is equal to the rectangles contained by the pairs of opposite sides. So, for the cyclic quadrilateral $APBC$, we have

$$AB.CP = BC.AP + AC.BP,$$

or, in the notation in Figure 1(i),

$$az = ax + ay.$$

Hence,

$$z = x + y;$$

that is, (1) follows quickly as an easy exercise on Ptolemy's Theorem. Thus, depending on our prior knowledge, van Schooten's Theorem might not detain us long.

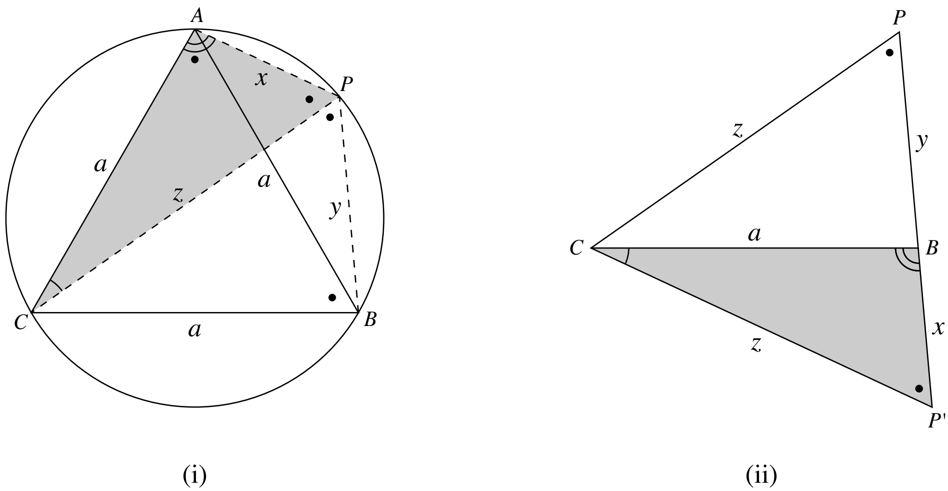


Figure 1: Van Schooten's Theorem

However, recollecting the historical context, it is also curious to note how (1) is suggested fairly readily in considering the geometry of a more celebrated problem, the challenge issued by Fermat in the early 1640s to those who, like Descartes, doubted his methods to find a point in a triangle that minimises the sum of the distances to the vertices. The late Folke Eriksson (1927–1999) contributed an informative and erudite account [9, (a)] on this subject to *Normat* in 1991 (see also [14, (a)] and [1, (c) §6.2]). Drawing on this article or otherwise, it is known that if P is the required point in the triangle $\triangle ABC'$, as in Figure 2(i), then in the first place the sides of the triangle subtend equal angles at P. But, secondly, as a construction for P, if an equilateral triangle $\triangle ABC$ is placed externally on the edge AB of $\triangle ABC'$, then P is the point in this triangle where the circumcircle of $\triangle ABC$ intersects CC', as shown in Figure 2(ii). Moreover, the length of CC' is the minimum sought by Fermat. Consequently, in the knowledge of this answer to Fermat's challenge, (1) follows immediately.

Van Schooten had been much closer personally to Descartes than to Fermat (compare [15, pp. 58, 25, 67, 56]). Indeed, Descartes in conversation with van Schooten had tended to put Fermat down, saying Fermat had discovered "many pretty, special things" in contrast to Descartes' own preference for general results with numerous applications. On the other hand, Fermat had been mortified that his earlier efforts went unmentioned when van Schooten issued a reconstruction of Apollonius' *Plane Loci* — van Schooten's student, Christian Huygens (1629–1695), even

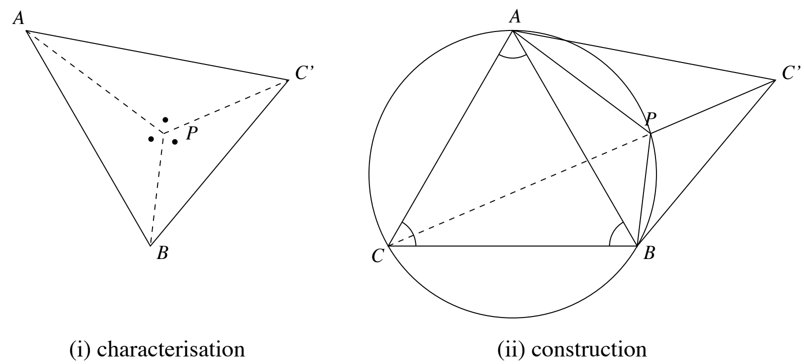


Figure 2: Fermat's Point

expressed the view that much of Fermat's restoration of this work was thin and perfunctory. But van Schooten is also known to have collected copies of Fermat's papers while in Paris. So, perhaps Fermat's challenge had not escaped van Schooten's notice when he advanced (1).

Certainly, (1) has all the appearance of a "pretty, special thing" of the sort that typified Fermat's work according to Descartes. Yet, as we shall see, it also has an aspect perhaps more satisfying to Descartes' tastes. For, a simple proof of this particular case of Ptolemy's Theorem pushes through, not only to establish Ptolemy's Theorem in full generality, but also to answering Fermat's challenge. Thus, van Schooten's modest case in (1) is actually equivalent to these seemingly more momentous results.

Some indication of this alternative line of development is pieced together in a couple of geometry textbooks [10] by Henry George Forder (1889–1981), at the start of the 1930s (the relevant passages are reproduced in Figure 7 by permission of the Forder Estate). Forder's approach has an affinity with the celebrated solution to Fermat's challenge to be found, for example, in [5, §1.8], but which had then appeared only a little earlier in a note [13, (a)] by Joseph Ehrenfried Hofmann (1900–1973). Van Schooten is mentioned neither by Forder nor in the current common reference [16, (a) §§1.5–6] for the link between Ptolemy's Theorem and Fermat's challenge (a fuller treatment of this link, including the equivalence of Fermat's principle of least time for light rays with Snell's law of diffraction, appears in a subsequent book [16, (b) §24] by the same author). However, Hofmann went on to write with great authority on the mathematics of both Fermat [13, (c)] and van Schooten [13, (b)], besides also returning more specifically to Fermat's challenge in [13, (d)]

Hofmann's demonstration might well go back further. For instance, Lothar Wolfgang von Schrutka, Edler von Rechtenstamm (1881–1945) presented something of a prototype [20, (a)] in 1914. Of late, there has been renewed interest in a generalisation of van Schooten's theorem noted in 1936 by Dimitrie Pompeiu (1873–1954) and to which we come in Section 5 (see, [3]; compare also [14, (c)] and [18]). But our starting point, in Section 2, is a recent variant of Forder's observation that appears in [11].

As it happens, the centenary volume [17] in which this note appears contains an essay [6] on Ptolemy's Theorem, although no connection is drawn between these contributions. More curiously in this respect, the volume includes a brief account [4] of a result of Ghiyath al-Din Jamshīd Mas'ud al-Kāshī (1390–1450) working at the apogee of late Arabic astronomical computation. Yet, despite al-Kāshī's result being firmly in Ptolemy's tradition of table making, it is neither presented nor recognised as a special case of Ptolemy's theorem, one indeed already anticipated by Euclid ($?\overline{325}$–$?\overline{265}$) in a proposition towards the end of *Data* [21], as we notice on re-encountering it in passing in Section 3. This fluctuating attention down the centuries, here encapsulated within a single volume, is suggestive alike of the workings of mathematical enquiry and of the historiography of mathematics.

Ptolemy presented his theorem on cyclic quadrilaterals in *Almagest I.10*. The instances where a side or a diagonal is a diameter of the circumcircle were instrumental in trigonometric computations for astronomical purposes. The classical proof in terms of similar triangles, touched on in Section 6, is rehearsed in such older textbooks as [12, 7], and anew with greater attention to visual presentation in [1, (a)]. But the Law of Cosines may be called in aid, as in [8], to develop algebraic expressions for the squares of the diagonals in terms of the sides, expressions that are themselves of considerable antiquity and from which Ptolemy's Theorem is an immediate consequence. This argument is turned around with the help of further visual aids in [1, (b)] to deduce the result on the diagonals given Ptolemy's Theorem.

2 The equilateral triangle

Now, the equilateral triangle is sufficiently special as to invite further thought in place of an appeal to another result, however well-known; perhaps ingenuity will come to our aid, if knowledge is deficient. The introduction of P on the circumcircle of $\triangle ABC$, cuts $APBC$ into two pieces $\triangle ACP$ and $\triangle BCP$ that fit together along CP. But, since the sides of $\triangle ABC$ are equal, following [11], it *looks* as though we can rotate the triangle $\triangle ACP$ about C so that AC matches up with BC. As suggested in Figure 1(ii), this seems to produce a new equilateral triangle $\triangle PP'C$. If so, then (1) will simply express the equality of the sides CP and PP' of this triangle.

We can achieve the effect of rotation by cutting off the triangle $\triangle ACP$ from $APBC$ and juxtaposing a triangle $\triangle BCP'$ congruent with it, thus conserving the angle at C:

$$\angle PCP' = \angle ACB = \pi/3.$$

Now, *Elements III.22* tells us that the opposite angles of a cyclic quadrilateral add to two right angles. In particular, then

$$\angle CBP' + \angle CBP = \angle PAC + \angle CBP = \pi.$$

So, this property in Figure 1(i) translates into the collinearity of P, B and P' in Figure 1(ii), ensuring that our surgery on the cyclic quadrilateral $APBC$ does result in a triangle, *viz* $\triangle PP'C$. We also have equal angles abounding, in view of *Elements III.21*, that angles subtended by a chord in the same arc of a circle are equal, and the fact that $\triangle ABC$ is equiangular:

$$\angle P'PC = \angle BPC = \angle BAC = \pi/3$$

and

$$\angle PP'C = \angle BP'C = \angle APC = \angle ABC = \pi/3.$$

Hence, our new triangle $\triangle PP'C$ is indeed equiangular and so equilateral.

Thus, our sense that dissecting $APBC$ into two pieces and interchanging them relative to one another to form another equilateral triangle proves well-founded, and we can push through to answer van Schooten's question. Put another way, the dissection demonstration *verifies* Ptolemy's Theorem in the special case of a cyclic quadrilateral $APBC$ where $\triangle ABC$ is equilateral.

3 The isosceles triangle

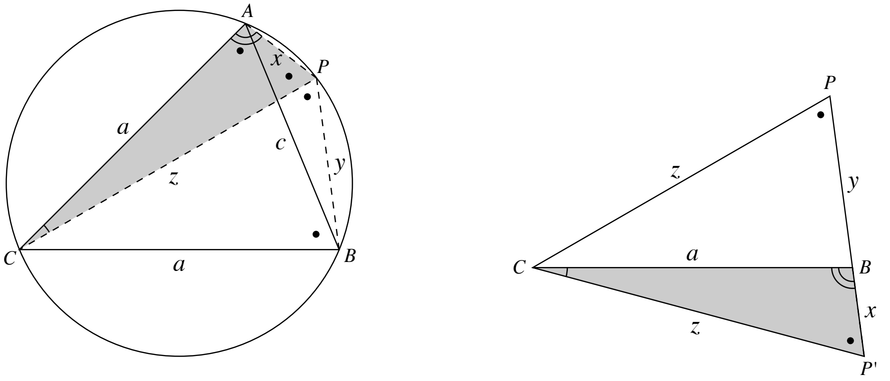
It is clear that, with results such as Ptolemy's Theorem, or even *Elements III.21* or *III.22*, we needs must know them in order to be in a position to use them; knowledge is a common hurdle that everyone has to surmount. But what can be a source of insight for one person can be an impediment to progress for another. After all, acquaintance with, say, Ptolemy's Theorem may not be lacking, but we still have to have some intuition to employ it in a given instance — sometimes knowing too much leaves us uncertain where to start.

What makes the dissection argument work so well with the equilateral triangle? It is important to understand the mechanics involved, if our intuition is not to become yet another stumbling block for those anxious about thinking through an answer to van Schooten's problem for themselves. Rather, approached in a spirit of critical research, the dissection demonstration might perhaps serve as a building block encouraging further investigation.

The crucial step in the dissection argument in the previous section is the initial one. Once we are assured that the sides AC and BC of $\triangle ABC$ are equal, everything else falls neatly into place, on account of standard results in circle geometry. But this assessment means that the demonstration will also work when $\triangle ABC$ is *isosceles* with equal sides AC and BC and so with equal base angles $\angle BAC$ and $\angle ABC$ (see Figure 2). For, the angle at C remains conserved; P, B and P' continue to be collinear; and we still have plenty of equal angles:

$$\angle P'PC = \angle BPC = \angle BAC; \quad \angle PP'C = \angle BP'C = \angle APC = \angle ABC. \tag{2}$$

Since $\angle ABC = \angle BAC$, we also have $\angle P'PC = \angle PP'C$. The upshot is that the transposition of the two pieces in our dissection produces another *isosceles* triangle



(i) (ii)

Figure 3: Isosceles triangle

$\triangle PP'C$ with equal sides PC and $P'C$. Moreover, in view of (2), the base angles in the isosceles triangle $\triangle ABC$ are equal to the base angles in the isosceles triangle $\triangle PP'C$, ensuring that these triangles are similar.

It follows that

$$AB : AC = PP' : PC; \tag{3}$$

that is, in the notation of Figure 2,

$$\frac{c}{a} = \frac{x+y}{z}. \tag{4}$$

Hence,

$$cz = ax + ay,$$

thereby verifying Ptolemy's Theorem in the case of a cyclic quadrilateral $APBC$ in which $\triangle ABC$ is isosceles with $AC = BC$.

We might note, as an instance of the isosceles case, the *right kite*, where in addition CP is a diameter of the circumcircle of $\triangle ABC$ so that $\angle CAP$ and $\angle CBP$ are both right angles. In this instance, we can also verify Ptolemy's Theorem by computing the area of the right kite in two ways, on the one hand because it consists of two congruent right triangles and on the other because the diagonals are at right angles.

The isosceles case is, in fact, the historical curiosity to which we alluded in the penultimate paragraph of Section 1. For, in one formulation, it predates Ptolemy's Theorem as such, having been given by Euclid in *Data*. But, in another, it reappears much later as the "fundamental theorem" of al-Kāshī, as it is called in [2].

4 The general triangle

This success encourages the further question: what happens if conditions are relaxed completely and $\triangle ABC$ is an *arbitrary* triangle? There is no problem in carrying out the surgery on $APBC$, deleting $\triangle ACP$ and adjoining a triangle $\triangle A'CP'$ congruent with it so that $A'C$ lies along BC (see Figure 3). But now there is no guarantee that A' will be coincident with B, as was, in effect, the case for equilateral and then isosceles triangles. However, as before, the angle at C is conserved; and at least we know that $A'P'$ is parallel to PB, because opposite angles of a cyclic quadrilateral add to two right angles. Thus, if we rescale $\triangle A'CP'$ by a factor of a/b to produce a similar triangle $\triangle BCP''$, then P, B and P'' will be collinear (see Figure 3(ii)). With this extra step, we have once more created a new triangle, $\triangle PP''C$.

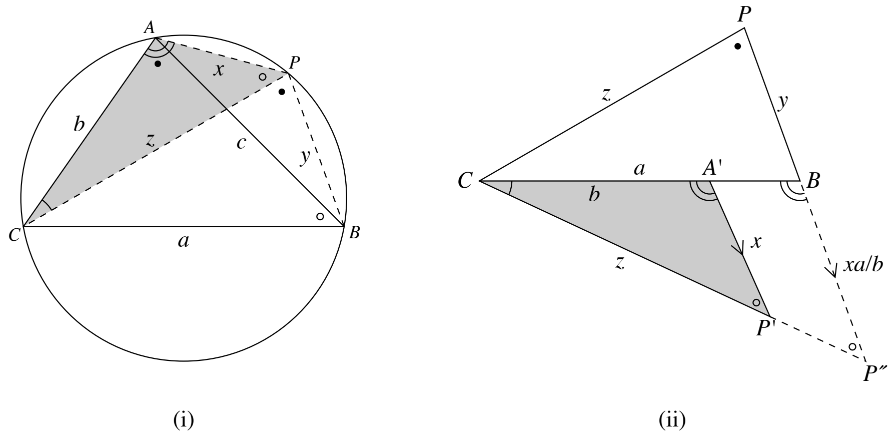


Figure 4: Ptolemy's Theorem

Now, the significant feature of $\triangle PP''C$ is that it inherits the angles of $\triangle ABC$, extending our findings in the special cases of equilateral triangles and then isosceles triangles. Not only is the angle at C in the former the same as that in the latter by construction, but also, corresponding to (2), we have

$$\angle P''PC = \angle BPC = \angle BAC$$

and

$$\angle PP''C = \angle BP''C = \angle A'P'C = \angle APC = \angle ABC.$$

Thus, $\triangle PP''C$ and $\triangle ABC$ are similar, since the three angles of one are pairwise equal to the three angles of the other.

In this more general case, instead of (3), we have

$$AB : AC = PP'' : PC.$$

Adopting the notation of Figure 3, this gives

$$\frac{c}{b} = (\frac{ax}{b} + y)/z,$$

which, rewritten as

$$cz = ax + by, \tag{5}$$

can then be recognised as a restatement of Ptolemy's Theorem. So, at this point, our discussion tips over, from an exploration of special cases, into an alternative proof of Ptolemy's Theorem, since now the cyclic quadrilateral $APBC$ is perfectly general. But had we known Ptolemy's Theorem, and been content in that knowledge, we might never have entered upon this different avenue of approach.

5 The general convex quadrilateral

Indeed, we now see that the only thing that goes wrong with the construction in Figure 3 when the quadrilateral $APBC$ is not necessarily cyclic is that P, B and P'' need not be aligned. However, triangles ΔABC and $\Delta PP'C$ remain similar since

$$\angle ACB = \angle PCP''; \quad AC : BC = PC : P''C, \tag{6}$$

by construction. So our previous argument goes through except that, instead of having PP'' explicitly, we must resort to the triangle inequality:

$$PP'' \leq PB + BP'',$$

with equality if and only if P, B and P'' are collinear, that is, if and only if $APBC$ is a cyclic quadrilateral, as before. Hence, as an extension of Ptolemy's Theorem, (5) becomes an inequality:

$$cz \leq ax + by, \tag{7}$$

with equality if and only if P lies on the arc AB of the circumcircle of ΔABC opposite C. It is exactly this extension that provides one approach among others to answer Fermat's challenge, as mentioned in Section 1. Since, as we saw there, the solution to the challenge implies (1), we now see that we can also find our way back from that solution to Ptolemy's Theorem and this extension as an inequality.

Returning to Figure 1, it is now apparent that, if P is not on the circumcircle of ΔABC, then P, B and P' are the vertices of a triangle with sides equal to x, y and z. This triangle has sometimes been named after Pompeiu, in honour of his work in the 1930s. So, putting the triangle PBP'' to work in the last step in our argument generalises Pompeiu's triangle into the bargain.

6 The right triangle

Of course, Ptolemy's Theorem contains within it Pythagoras' Theorem that, for a right triangle, the square on the hypotenuse is equal in area to the sum of the squares on the legs, as can be seen by restricting the cyclic quadrilateral to be a rectangle. On the other hand, the classical proof of Ptolemy's Theorem procedes rather as an outworking of Euclid's demonstration of *Elements VI.31* extending Pythagoras's Theorem from squares to similar figures, in particular similar triangles, similarly situated on the edges of the right triangle, starting with the case in Figure 5(i) where the cyclic quadrilateral $APBC$ is a rectangle. In this case, let L be the foot of the perpendicular from C onto AB. Then, following Euclid, the triangles $\triangle ACL$ and $\triangle CBL$ are both similar to $\triangle ABC$ and so respectively to $\triangle PCB$ and $\triangle CPA$. From these pairings, we infer that

$$AL.CP = AC.BP, \quad LB.CP = CB.AP,$$

which taken together yield

$$AB.CP = (AL + LB).CP = AC.BP + CB.AP \tag{8}$$

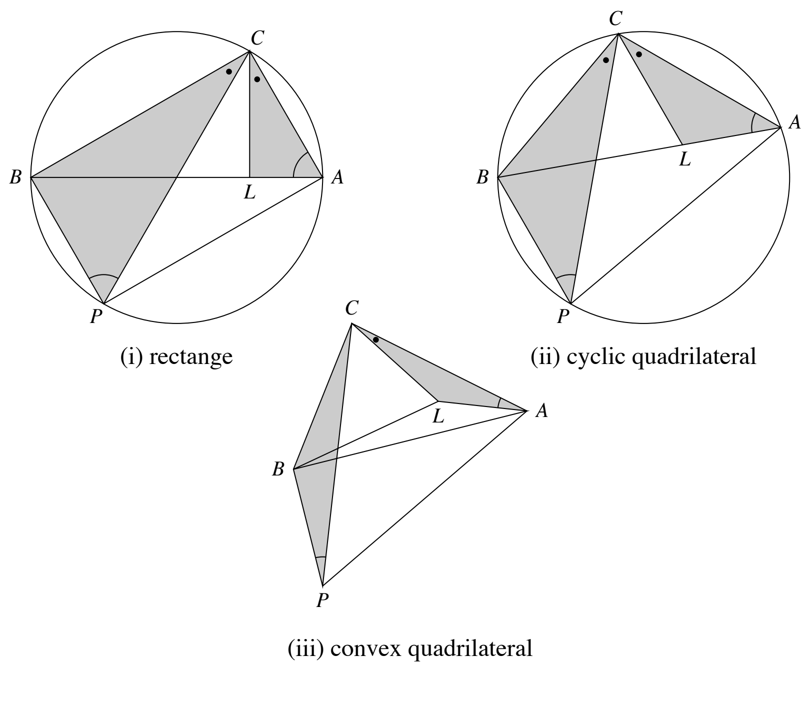


Figure 5: From Pythagoras to Ptolemy

At this juncture, (8) is no more than a restatement of Pythagoras' Theorem, as opposite sides of the rectangle are equal, $AB = CP$, $AC = BP$, and so are the diagonals, $BC = AP$. The trick in turning (8) into Ptolmey's Theorem is to allow L to vary on AB so as to retain one pairing of the similar triangles, say, $\triangle ACL$ with $\triangle PCB$ (suggested by the shading in Figure 5(ii)). For, this ensures that

$$\angle BCL = \angle BCP + \angle PCL = \angle PCL + \angle LCA = \angle PCA, \tag{9}$$

while $\angle CBL = \angle CPA$ by *Elements III.21*. Thus, $\triangle CBL$ and $\triangle CPA$, having pairwise equal angles, remain similar. Hence, we recapture (8) for the general cyclic quadrilateral, and with it a motivated presentation of the classical proof of Ptolemy's theorem. In the spirit of our earlier argument using a single pivot with scaling, each pair of similar triangles can be seen as a separate pivot with scaling, although this might seem artificial without our motivation (in [1, (a)], the argument precedes a separate treatment of Fermat's point).

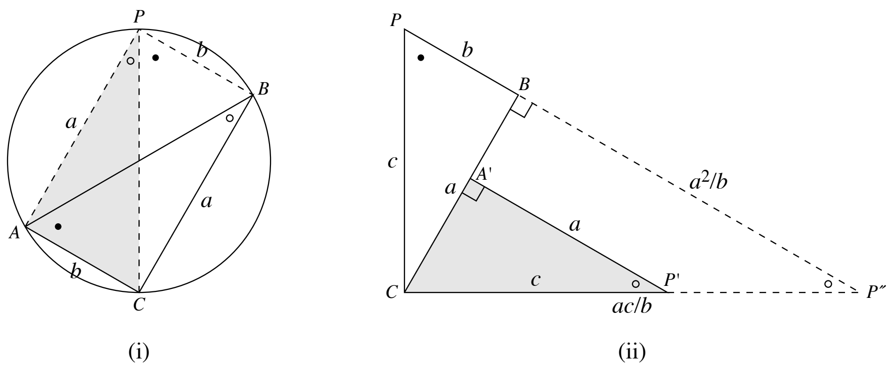


Figure 6: From van Schooten to Pythagoras?

To cap this argument for a general convex quadrilateral after the manner of Section 5, we can take L so that $\triangle ACL$ and $\triangle PCB$ are similar (see Figure 5(iii)). We shall still have, in addition to (9),

$$AC.PC = LC.BC.$$

This combination is enough to ensure that $\triangle CBL$ and $\triangle CPB$ are similar (compare (6)). However, L is no longer restricted to lie on AB, so the triangle inequality turns (8) into (7) (compare [1, §6.1]).

It is of further interest to see what becomes of our proof in Section 4 when $APBC$ is the rectangle in Figure 6(i). The result, shown in Figure 6(ii), is that $\triangle PP''C$ is a right triangle similar to $\triangle ABC$, with BC the altitude at C dividing $\triangle PP''C$ into two further similar right triangles. This arrangement of similar right triangles returns us, in effect, to *Elements VI.31* (compare Figure 5(i)). At least in theory, we could start with the figure for *VI.31* and imagine ourselves reversing the steps to devise the alternative proof of Ptolemy's Theorem by dissection and rescaling. However, the right triangle seems a less plausible starting point for this than van Schooten's equilateral triangle, perhaps because it requires a greater sense of what to do next at each stage. Some doors are trap doors.

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length (Example 16, p. 50). On the sides of $ABCD$ draw segments congruent to those of the sides of $A'B'C'D'$. Then use No. 3.

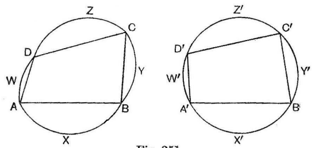

Fig. 251

Or, use the formula for the area of any quadrilateral, with sides a, b, c, d:

Sq. of area $= (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha$,

where $2s = a+b+c+d$, 2α = sum of two opposite angles.

5. *Fermat's Problem. Given a triangle ABC, to find a point F within it such that $AF + BF + CF$ is a maximum.*

Construction. Draw an equilateral $\triangle BCD$ on BC, on the side opposite to A.

Then AD cuts the circumcircle of BCD in the required point F.

Proof. $DF = BF + CF$. (By Ptolemy's Theorem on $BFCD$, or by *S.G.*, p. 132.)

$\therefore AF + BF + CF = AF + DF = AD.$

But if X is any point in the plane, then by the converse of Ptolemy's Theorem applied to the quadrilateral $BXCD$,

$$BX.DC + CX.DB \geqq DX.BC.$$

$$\therefore BX + CX \geqq DX.$$

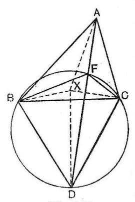

Fig. 252

$\therefore AX + BX + CX \geqq AX + DX$, equality holding only when X is on the circle.

But $AX + DX \geqq AD$, equality holding only when X is on the interval AD.

Hence $AX + BX + CX > AD$, unless X is at F.

Note. The argument fails if any angle of the given triangle is greater than 120°. In that case the point required is at the vertex of this angle. If equilateral triangles be described on the sides of any triangle ABC and outside it, the lines from A, B, C to the opposite

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vertices of the equilateral triangles meet in the same point, and the sides of the triangles subtend angles of 120° or 60° there. If this point falls inside the triangle, it is the point required.

1. ABC is an acute-angled triangle, D a given point on BC. Find points E, F on BA, CA resp. so that the perimeter of $\triangle DEF$ is a minimum.

2. ABC is an acute-angled triangle. Find points D, E, F on BC, CA, AB resp. so that the perimeter of $\triangle DEF$ is a minimum.

3. Find points A, B, one on each of two given circles, so that AB has a given direction and the interval AB is (i) a maximum, (ii) a minimum.

4. P is a point inside an angle BAC. Through P draw a line cutting AB, AC resp. in Q, R so that the perimeter of $\triangle QAR$ is a minimum.

5. From a point outside a given circle, centre O, draw a line cutting the circle on X, Y so that $\triangle XOY$ has maximum area.

6. If a cyclic quadrilateral be given, then, in general, two other cyclic quadrilaterals (not congruent to the first, or to each other), can be drawn with sides of the same length as those of the given quadrilateral. All these quadrilaterals have the same circumradius and area. Thus, of the quadrilaterals whose sides (in any order) are of given lengths, there are three of equal area, whose area is greater than any other such quadrilateral; a quadrilateral of *minimum* area, with sides of given length, is non-convex, and if two of its sides be reflected in the exterior diagonal, we obtain a quadrilateral of maximum area. How many quadrilaterals of minimum area are there?

§ 2. The Problem of Apollonius.

LEMMA 1. *If two non-concentric circles $\mathcal{K}_1$, $\mathcal{K}_2$ touch two other non-concentric circles $\mathcal{C}_1$, $\mathcal{C}_2$ at A, B and at C, D respectively, and the contacts at C and D are of like or unlike type according as those at A and B are, then AB and CD cut in a centre of similitude of the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ or they are parallel to the line of centres of these circles* (Fig. 253).

[Contacts are of '*like type*' when both are external or both internal. Similarly, two centres of similitude are of '*like type*' when both are external or both internal.]

Proof. C, D are like or unlike centres of similitude of the pairs of circles touching at C, D, according as A, B are like

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8. (*a*) $ABCD$ is a parallelogram. A circle through A, B cuts AD, BC in P, Q resp.; a circle through C, D cuts them in R, S resp. Prove $PQ \parallel RS$.

(*b*) Prove this also when $ABCD$ is a trapezium with $AB \parallel CD$.

9. ABC is an isosceles triangle; $AB = AC$; a circle through B, C cuts AB, AC resp. in X, Y. Prove $AX = AY$.

10. $ABCD$ is a quadrilateral; $A\hat{B}C = B\hat{A}D$; X is on the side AB; the circles AXD, BXC cut CD again in Y, Z resp. Prove $XY = XZ$.

11. Two circles cut in A, B; X is a point on the chord AB produced beyond B. A line through A cuts the circles in C, Y resp. on opposite sides of A; XC, XY cut the circles again in D, Z. Prove

$$Z\hat{B}D + Z\hat{X}D = 2 \text{ rt. } \angle\text{s}.$$

12. Two circles $XYQP$, $ZWSR$ are external to one another; one line cuts them in points in order X, Y, Z, W; another line cuts them in points in order P, Q, R, S; QY, RZ cut in A; XP, WS cut in B. Prove that the angles at A equal those at B.

13. Two circles cut in X, Y. A line through X cuts them in A, B resp.; AP, BQ are parallel chords, one of each circle. Prove that P, Y, Q are collinear.

EXAMPLE 26. *ABC is an equilateral triangle inscribed in a circle. P is a point on the minor arc BC. Prove*

$$PA = PB + PC.$$

Proof. Produce CP beyond P to D, making $PD = PB$.

Since $B\hat{A}C = 60°$, (angle in equil. $\triangle$).

$\therefore B\hat{P}D = 60°$, (ext. $\angle$ of $ABPC$ = int. opp. $\angle$).

But $PB = PD$, (const.).

$\therefore \triangle BPD$ is equilateral.

In $\triangle$s ABP, CBD,

$BA = BC$, ($\because \triangle ABC$ is equil. $\triangle$).

$BP = BD$, ($\because \triangle BPD$ is equil. $\triangle$).

$A\hat{B}P = C\hat{B}D$, (each $= C\hat{B}P + 60°$).

$\therefore \triangle$s ABP, CBD are congruent, (two sides, incl. $\angle$).

$\therefore PA = DC$. But $DC = PC + PB$, (const.).

$\therefore PA = PC + PB$. Q.E.D.

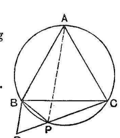

Fig. 178

14. P is on the bisector of angle BAC; circles APB, ACP cut BC in Q, R resp. Prove $PQ = PR$.

15. Two circles $ABQP$ and $ABYX$ cut in A, B. A line is drawn through A, and a line through B. These cut one circle in P, Q, the other in X, Y. If $PQ = XY$, prove $PX = QY$.

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16. $ABCDEF$ is a hexagon inscribed in a circle. Prove that the sum of the angles at A, C, E equals four right angles.

17. The altitudes AP, BQ of a triangle ABC cut in H. Prove

$$A\hat{H}Q = B\hat{C}Q, \quad C\hat{P}Q = B\hat{A}Q.$$

18. Show how to construct a triangle ABC, given sides AC, BC and the altitude from A, all in length only.

19. Two triangles with bases, areas, and vertical angles equal, are congruent.

20. AP, BQ, CR are altitudes of $\triangle ABC$; L is the mid-point of BC. Prove $R\hat{Q}L = Q\hat{R}L = B\hat{A}C$.

21. CD is a chord of a circle perpendicular to the diameter AB; a line through A cuts the circle in X, and CD, produced beyond D, in Y. Prove $A\hat{X}C = D\hat{X}Y$.

22. AB, CD are perpendicular chords of a circle, cutting in E; the perpendicular from C to AD cuts AB in X. Prove $XE = EB$.

*23. AB is a common chord of two circles, centres C and D. The line PAX cuts the circles in P, X resp. Prove $C\hat{P}B = D\hat{X}B$. (Draw a careful figure.)

*24. Two circles cut in A, B; X is on one circle, centre O; XA and XB cut the other circle in C, D resp. Prove $XO \perp CD$ (Fig. 179.)

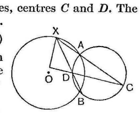

Fig. 179

*25. P is any point inside $\triangle ABC$; D, E, F are the feet of the perpendiculars from P to BC, CA, AB resp. Prove

$$B\hat{P}C - B\hat{A}C = E\hat{D}F.$$

*26. Two circles cut in A, B. Through A draw a line cutting the circles in P, X, so that $X\hat{B}A = P\hat{B}A$.

Figure 7: Facsimiles from Henry Forder's books *School Geometry* (1930) and *Higher Course Geometry* (1931), reproduced by permission of the Forder Estate