What is the most important aspect of Smale's horseshoe?

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This paper is based on a project I did in the spring 2010 at Roskilde University as part of my Master's degree. The aim of the project was to investigate the influence of Smales horseshoe on the developement of the theory of dynamical systems. I analysed three cases where Smales horseshoe had been included in the study of both the Hénon mapping and the restricted three body problem. From the analysis of these cases you get the distinct impression, that the most important aspect of the horseshoe is that it is topologically conjugate to a shift autophism and so can be used as a translator between the shift automorphism and another dynamical system which has a horseshoe imbedded in its dynamics. In contrast to this Stephen Smale himself states in a paper from 1980 [Smale, 1980] that the most important aspect of the horseshoe is not the connection to the shift automorphisms, but the fact that the horseshoe is structurally stable.

The aim of this paper is to promote a point of view giving a possible explationation of this disagreement. The structure of the paper is as follows: In the first section Smales horseshoe is presented. In the second section the three cases that I have analysed are briefly introduced. In the third section a possible explaination is given for why Smale in the paper from 1980 wievs the structural stability of the horseshoe as the most important aspect of it. In the fourth section a possible reason for the disagremeent between the cases and Smales paper will be discussed.

Smales horseshoe

In this section the main idea behind Smales horseshoe is introduced. I am only going to write about Smales original version of the horseshoe mapping, which he presented in two article from 1965 and 1967 respectively [Smale, 1965], [Smale, 1967]. At the end of this section I will comment on the relation between this version of the horseshoe and the three cases that I have analysed.

Let g be a diffeomorphism of a subset Q of $\mathbb{R}^2$ to $\mathbb{R}^2$. Smale defines Q to be the square $Q = \{(x, y)||x| \leq 1, |y| \leq 1\}$ in $\mathbb{R}^2$ [Smale, 1967, p. 770]. The mapping g contracts the square Q in the vertical direction, expands it in the horizontal direction and folds it back over itself, so that $g(Q)$ is shaped like a horseshoe that lies over Q as in figure 1. In other words g is a mapping that maps Q onto the

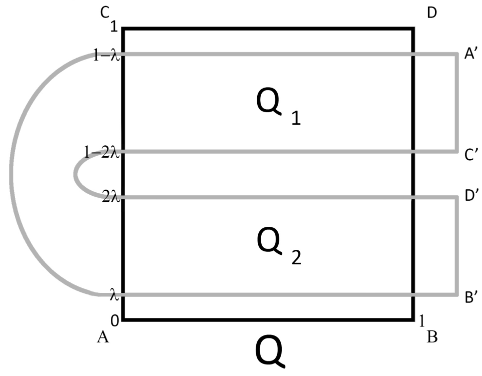

Figure 1: The square Q and the horseshoe shaped image of Q under g.

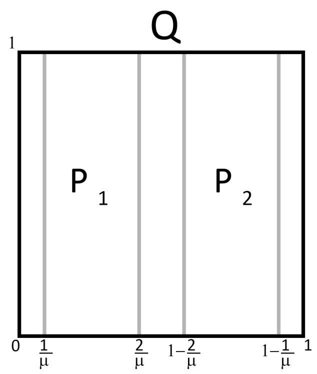

Figure 2: The square Q and the set $P_1 \cup P_2 = g^{-1}(g(Q) \cap Q)$.

horseshoe shaped region of figure 1 in such a way that $g(A) = A'$, $g(B) = B'$, $g(C) = C'$ and $g(D) = D'$. From figure 1 it is easy to see that the intersection $Q \cap g(Q)$ consists of the two rectangles Q_1 and Q_2. According to Smale any mapping that has the following two properties can be used as the mapping g [Smale, 1967, p. 771]:

1. The mapping g is a diffeomorphism of Q onto the horseshoe shaped region of figure 1, that is defined so that $g(A) = A'$, $g(B) = B'$, $g(C) = C'$ and $g(D) = D'$.

2. On each of the components P_1 and P_2 of the set $g^{-1}(Q \cap g(Q))$ (see figure 2) g is an affine mapping.

According to Smale a consequence of the second property is that P_1 and P_2 will be as in figure 2 and $g(P_1) = Q_1$ and $g(P_2) = Q_2$ [Smale, 1967, p. 771]. The mapping g is what we will call a horseshoe map.

If g is used repeatedly on Q more and more of the points that starts in Q is mapped outside of Q. What is interesting regarding Smales horseshoe is the set of points which stays in Q under infinitely many both forwards and backwards iterations with g. This set of points is denoted Λ and is given by [Smale, 1967, p. 771]

$$\Lambda = \bigcap_{m \in \mathbb{Z}} g^m(Q^{(m)}) \tag{1}$$

where $Q^{(m)}$ is the intersection of Q and $g^{m-1}(Q)$ when $m > 0$, $Q^0 = Q$ and $Q^{(m)} = g^m(Q^{(m+1)})$ when $m < 0$. It is easy to see that Λ is invariant under g. Futhermore on Λ the map g is topologically conjugate to the shift automorphism $\alpha : X_S \mapsto X_S$, where S is a set that contains two elements and $X_S = S^{\mathbb{Z}}$ [Smale, 1967, p. 771]. The elements in X_S can be written as biinfinite strings. The action of α on a string is that it shifts all entries one place to the right. The shift automorphism possesses a countably infinite set of periodic orbits, a uncountable set of aperiodic orbits and a dense orbit, and since the shift automorphism is topologically conjugate to the horseshoe, the horseshoe too has these properties [Wiggins, 1990, p. 436].

In his article from 1965 Smale shows, that the horseshoe map g is structurally stable [Smale, 1965, p. 63 and p. 74-77]. This means that in the space of diffeomorphisms on $\mathbb{R}^2$ there is an open neighborhood of g such that every diffeomorphism in that neighborhood is topologically conjugate to g.

As mentioned above the horseshoe which I have presented in this section is Smales original version of the horseshoe. When it comes to the application of Smales horseshoe in the study of other dynamical systems, this version is not very useful. This is because of the two requirements to the horseshoe map that Smale outlines and that many dynamical systems fail to live up to. Indeed the version of the horseshoe that is included in two of my three cases is not Smales original version, but a more generalised version, that is descriped e.g. by Moser [Moser, 1973]. When it comes to the connection between the horseshoe and the shift automorphism and the structural stability of the horseshoe however there is no difference at all between Smales original version of the horseshoe and the generalised version. Therefore I will not say anymore about the different versions of the horseshoe.

The three cases

The aim of this section is to give a brief presentation of the three cases and to explain how Smales horseshoe is included in each of them.

The first case is from a book called *Stable and random motions in dynamical systems* written by Jürgen Moser in 1973 [Moser, 1973]. In the third chapter of the book Moser presents a proof of a fascinating theorem about the dynamics in the restricted three body problem. The version of the restricted three body problem which he is working with consists of two particles, m_1 and m_2, of equal masses that are moving along two identical elliptic orbits placed symmetrically around the center of mass of the two paticles. These orbits are contained in a fixed plane s in space. The third particle, m_3, has a mass so small compared to the masses of m_1 and m_2 that it can be ignored. This means that m_1 and m_2 are acting on m_3, but m_3 is not acting on m_1 and m_2. The particle m_3 is moving along a line, which is perpendicular to s and which passes through the center of mass of m_1 and m_2. The restricted three body problem is to determine the motion of m_3 when the initial

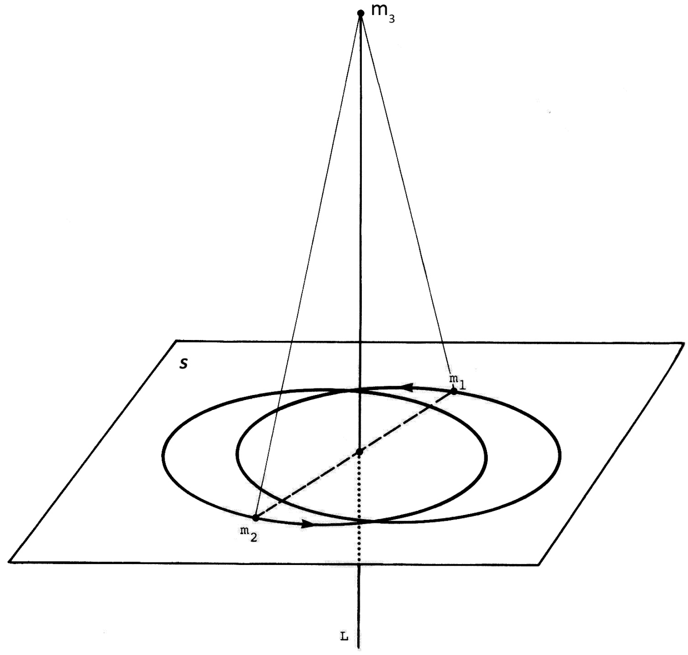

Figure 3: The restricted three body problem. The figure is borrowed from [Moser, 1973, p. 84] and modified by me.

conditions of the three particles are known. The scenario of the restricted three body problem is shown in figure 3.

The theorem which Moser proves is concerned with the possible motions of the third particle in the restricted three body problem. Let $z(t)$ denote a particular such motion that intersects the plane s infinitely many times. These intersections occur at the times given by the biinfinite sequence $\{t_k\}$. Now choose the units in such a way that the time it takes m_1 and m_2 to complete one revolution in their orbits equals 2π. Then the number of complete revolutions that m_1 and m_2 performs between two successive intersections between m_3 and s is given by

$$\sigma_k = \left[\frac{t_{k+1} - t_k}{2\pi} \right] \tag{2}$$

where $[x]$ denotes the biggest integer that is less than or equal to x [Moser, 1973, p. 85]. Clearly any motion of m_3 having an infinite sequence of passing times of s in both future and past corresponds to a biinfinite sequence of integers $\{\sigma_k\}$, such that the numbers of the sequence describe the number of complete revolutions that m_1 and m_2 performs between two successive passing times of s. Mosers theorem states the opposite, namely that given a sufficiently small eccentricity $\epsilon > 0$ of the orbits of m_1 and m_2 there is a integer $n = n(\epsilon)$ for which for every sequence $\{\sigma_k\}$ that satisfies the requirement that $\sigma_k \geq n$, there is a motion of m_3 such that the numbers of the sequence describe the number of complete revolutions which m_1 and m_2 performs between two successive passing times of s of that motion [Moser, 1973, p. 85]. Moser proves this theorem by finding a horseshoe imbedded in the dynamical system that describes the motion of m_3. Having found this horseshoe he uses the connection between the horseshoe and the shift automorphisms to argue for the theorem. For Moser the horseshoe is included as a tool by means of which he can prove his theorem and in this context the most important aspect of the horseshoe is its connection with the shift automorphisms.

The second case is from an article called *On the Hénon Transformation* written by James H. Curry in 1979 [Curry, 1979]. In this article Curry presents the results of some numerical studies of the Hénon mapping which Hénon presented in an article written in 1976 [Hénon, 1976]. The aim of some of these numerical studies was to produce evidence that there is a Cantor set in the trapping region of the state space of the Hénon mapping with the parameter values used by Hénon. The trapping region is a region of the state space that is mapped to itself under the Hénon mapping [Hénon, 1976, p. 75-76]. This means that trajectories which enters this region is trapped inside it. Curry argues for the existence a Cantor set in the trapping region of the state space of the Hénon mapping by finding a strong indication of the existence of a horseshoe in the trapping region. The reason that the existence of a horseshoe implies the existence of a Cantor set is that the subset Λ of the domain of the horseshoe map g, on which g is topologically conjugate to a shift automorphism, is homeomorphic to a Cantor set. This follows from the fact that the domain of the shift automorphism is homeomorphic to a Cantor set [Smale, 1967, p. 770]. Like Moser Curry too uses the horseshoe as a tool by means of which he provides evidence, that there is a Cantor set in the trapping region of the Hénon mapping. Even though Curry is not using the connection between the horseshoe and the shift automorphism as explicitly as Moser, he is using the horseshoe in a way which he would not have been able to if the connection had not been there.

The third case is from an article called *Shift Automorphisms in the Hénon Mapping* written by Robert Devaney and Zbigneiw Nitecki in 1979 [Devaney and Nitecki, 1979]. In this article Devaney and Nitecki presents a proof of a theorem that among other things states, that it is possible to choose the parameters of the Hénon mapping in such a way, that the Hénon mapping restricted to a subset of its domain is topologically conjugate to the shift automorphism. This statement is proved by finding a horseshoe imbedded in the Hénon mapping. Unlike Moser and Curry Devaney and Nitecki do not use the existence of a horseshoe and hence a shift automorphism in the Hénon mapping to prove another result. The finding of a horseshoe is for them not a tool, but an end. Like Moser and Curry however Devaney and Nitecki emphasises the connection between the horseshoe and the shift automorphism.

In all of the three cases Smales horseshoe is included as either a tool or an end of a proof. Furthermore, even though the aims of the three cases are different they are all emphasising the connection between the horseshoe and the shift automorphisms more than they emphasise any other aspect of the horseshoe, thus leaving the impression that this aspect of the horseshoe is the most important one.

Smales paper from 1980

Smales paper from 1980 is called *On how I got started in dynamical systems* and consists of a rather informal description of his early work in dynamical systems, which among other things includes the discovery of the horseshoe. As mentioned in the beginning Smale does not in his 1980 paper consider the connection between the horseshoe and the shift automorphism to be the most important aspect of the horseshoe. Actually he does not even mention this connection. Instead he more or less explicitly states that the most important aspect of the horseshoe is its structural stability. I believe, that the reason for this is that the 1980 paper mainly is concerned with the development which led to the discovery of the horseshoe. The reason for the view of the 1980 paper is in other words the context of the paper.

In 1956 Smale finished his thesis in topology, and according to himself he considered himself mainly a topologist [Smale, 1980, p. 150]. Nevertheless he became more and more interested in ordinary differential equations and dynamical systems, and in 1959 he wrote his first paper on dynamical systems [Smale, 1960]. This paper was about Morse inequalities for a class of dynamical systems, that later has been named Morse-Smale dynamical systems, and which among other things is characterised by having a finite number of fixed points and periodic orbits. In the paper Smale conjectured, that the Morse-Smale dynamical systems form an open dense set in the space of all ordinary differential equations [Smale, 1960, p. 43], [Smale, 1980, p. 148].

After his article on the Morse-Smale systems was published in 1960 Smale recieved a letter from Norman Levison. In this letter Levinson pointed out, that Smales conjecture about the Morse-Smale systems being open and dense in the space of all ordinary differential equations could not be true, since an earlier paper of Levinson contained a counterexample to it. Levinsons paper was a simplification of the work on the Van der Pol equation done by Cartwright and Littlewood [Smale, 1980, p. 149], [Guckenheimer, 1980, p. 987]. In an article from 1998 Smale says, that he had to work hard to translate Levinsons counterexample into a more geometric form before he was convinced that Levinson was right and that his own conjecture was wrong [Smale, 1998, p. 41]. The geometric version of Levinsons counterexample which came out of Smales struggle was the horseshoe. The horseshoe is a counterexample to Smales early conjecture because of its structural stability, which as mentioned above means that there is an open neighborhood of the horseshoe mapping such that every mapping in this neighborhood is topologically conjugate to the horseshoe. Since the horseshoe possesses a countable infinity of periodic orbits, Smales early conjecture thus can not be right.

It would be fair to say, that one of the main developments that Smale describes in his 1980 paper is the development of his own understanding of the space consisting of all ordinary differential equations. In this development the discovery of the horseshoe played a central role, because it was this discovery that made Smale realise, that his early comprehension of the structure of this space was wrong. And regarding this realisation the structural stability of the horseshoe is completely indispensable.

Conclusion

Why is it that in the three cases the most important aspect of the horseshoe is its connection to the shift automorphisms, whereas in the 1980 paper Smale emphasises the structural stability of the horseshoe as the most important aspect? If one considers the way the horseshoe is included in the three cases, it seems reasonable to say, that they do not emphasise the structural stability of the horseshoe, because this aspect is not important to their work. The structural stability can not be used to make their proofs any easier, prettier or more ingenious. The connection between the horseshoe and the shift automorphisms on the other hand can be used to do this, which is especially clear from the cases of Moser and Curry. This seemes to me to be a possible explanation of why the three cases emphasises the connection between the horseshoe and the shiftautomorphism instead of the structural stability of the horseshoe.

Now let us focus our attention on the 1980 paper of Smale. The reason why this paper nominates the structural stability of the horseshoe to be the most important aspect of the horseshoe is, that this aspect played a decisive role in the development of Smales understanding of the space of all ordinary differential equations, which is one of the main themes in the paper. The connection between the horseshoe and the shift automorphism on the other hand was of no real importance to this development, and thus there is no reason for mentioning it in this paper.

In the light of these reflections a possible explanation of the disagreement between the three cases and the 1980 paper of Smale is that the cases and the paper speak about the horseshoe from two different contexts - and seen from these two contexts the most important aspect of Smales horseshoe is not the same. According to Tinne Hoff Kjeldsen the context in which a mathematical result or concept is considered often has a great impact on how the result or concept is regarded [Kjeldsen, 2009, p. 110]. Therefore I find my explanation of the disagreement not only possible, but also plausible.

From the above discussion it must be concluded, that the question of the title of this paper can not be given an unequivocal answer. The reason of the disagreement between the three cases and the 1980 paper is not that one of them is wrong, but instead that they are looking at Smales horseshoe from different contexts. This reflection must entail, that the answer to the question of what the most important aspect of Smales horseshoe is will depend on the context that one considers the horseshoe from. This point becomes especially clear, if we look at the article called *Finding a Horseshoe on the Beaches of Rio* that Smale wrote in 1998 [Smale, 1998]. In this article Smales emphasises the chaotic behaviour of the horseshoe more than any other aspect of the horseshoe. This aspect though is not mentioned a singel time neither in the three cases nor in the 1980 paper, even though this paper too is written by Smale. That Smale himself in two different papers implicitly nominates two different aspects of the horseshoe to be the most important one clearly shows, that you can not say what the most important aspect the horseshoe is. The only thing which you can say is, what the most important aspect is seen from the context that you are working in.

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