Bisecting lines of convex regions

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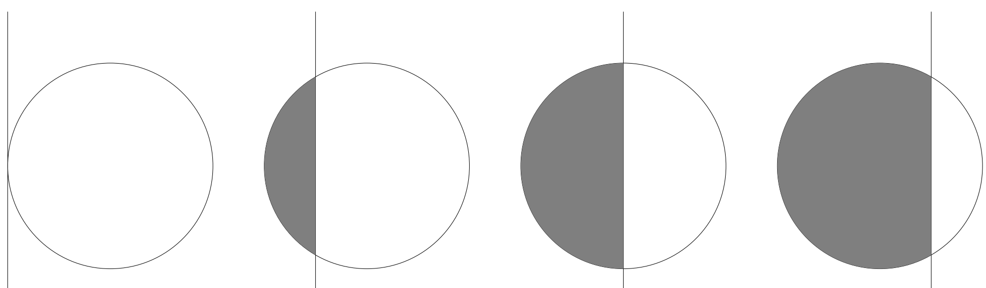
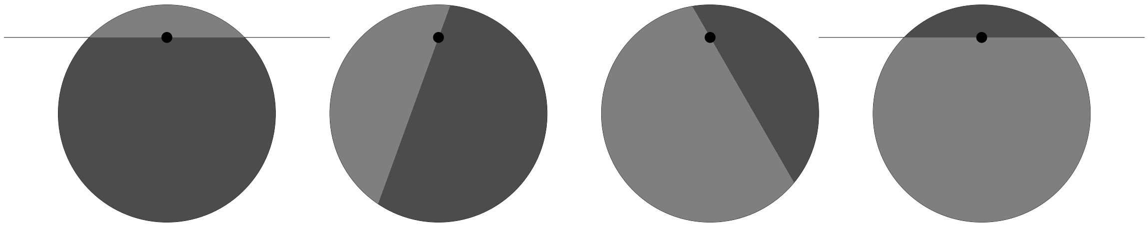
Introduction

Let us to fix ideas consider compact convex domains Ω in the plane (although much of what is going to be said will also hold in a more general case) and furthermore let us call a line bisecting if it divides Ω into two parts of equal area.

It is easy to see that there are plenty of bisecting lines. In fact if we fix the direction of lines there will be a unique line among them which is bisecting. We simply translate a given line and see what is say on the left of it in Ω. The area of which will vary monotonically and continuously as we move to the right, starting with zero and ending up with the area of Ω. Somewhere in between we will have area $\frac{1}{2}\mu(\Omega)$. In fact all parallel lines can be written under the form $ax + by = c$ fixing $(a, b) \neq 0$. And we can define $\Omega_c = \{(x, y) \in \Omega : ax + by < c\}$. The function $\mu(c) = \mu(\Omega_c)$ is strictly increasing and continuous varying from 0 to $\mu(\Omega_c)$.



A similar argument shows that through each point there goes a bisecting line. We simply rotate the line through the point. It will divide Ω into a left and right part say. When we have rotated a half turn, those two parts have been switched. So if we consider the difference in area, which likewise varies continuously its value have changed signs, thus somewhere between it has to have been zero.



Note that if the point lies outside Ω there will be a unique bisecting line through it. On the other hand if it lies inside there may be many bisecting lines through it.

A natural question to ask is whether all the bisecting lines go through a single point. This is obviously the case for a circle, or more generally an ellipse or a rectangle. Medians are bisecting lines for triangles, and medians intersect in a single point. Such a common point to all the bisecting lines would be a natural candidate for being the true center of the region. However, most regions do not have a center in fact we can easily establish the following.

Proposition: *A point p is the center of a convex region Ω iff it is on the midpoint of each segment $L \cap \Omega$ (a so called bisecting segment) where L is a bisecting line.*

PROOF: Consider a radius vector through the point p going to the boundary of Ω. Those are parametrized by the angles (θ) they make with the $x-$axis. We can then define a function $l(\theta)$ given by their lengths. As the radius vector rotates in the positive direction (θ increases) we can define the sets $\Omega_\theta^+, \Omega_\theta^-$ of the points in Ω ahead or behind the line. If we let $\mu(\theta) = \mu(\Omega_\theta^+)$ we see that

$$\frac{d\mu(\theta)}{d\theta} = \frac{1}{2}(l(\theta) - l(\theta + \pi))$$

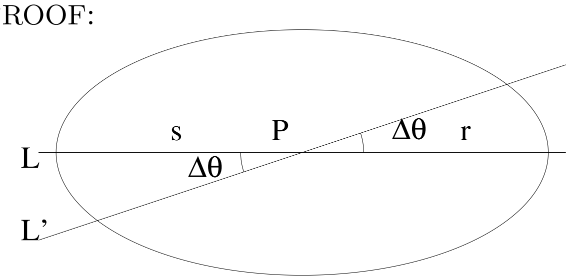
Thus the function μ is constant iff $l(\theta) = l(\theta + \pi)$

Thus a region with a center in this sense, would be invariant under reflection in that center. In other words if the center is translated to the origin, the region would be invariant under $(x, y) \mapsto (-x, -y)$. Clearly most regions do not have points with this property, in particular triangles do not, because the only candidate for such a point would be its center of gravity (the intersection of any two of its medians) and for such it is easy to check that it is not the case. But what do we have instead?

So let us fix a bisecting line L and consider all other bisecting lines M. They will all intersect L in a varying point LM which always will lie inside Ω from a remark made above. When M approaches L one will expect a limiting intersection point, but *a priori* it is not entirely clear that this will not depend on whether we approach in the positive or negative direction. But it is actually easy to see that the self-intersection will be the midpoint of the bisecting segment.

Proposition: *The self-intersection point of a bisecting segment is its mid-point.*

PROOF:



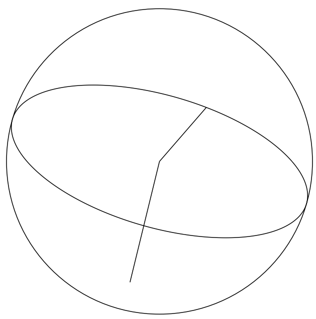
Consider two bisecting segments L, L' which are close and intersect in P. They cut the region into four parts, two and two non-adjecent. The non-adjecent parts have the same areas, in particular the two areas defined by the acute angle. Those areas will be better and better approximated by $\frac{1}{2}r^2\Delta\theta$ and $\frac{1}{2}s^2\Delta\theta$ as $\Delta\theta \to 0$ from which we conclude that in the limit as L' approaches L we get $r = s$

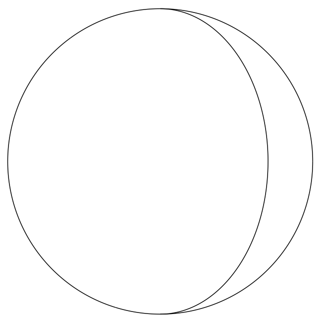
The total image of all the intersection points will make up a connected segment $\hat{L}$ of L contained in the interior of Ω. As we vary L there will be a region traced out inside Ω and whose boundary will be made up of the boundary points of the $\hat{L}$. That region we will call the bisecting image and denote by $B(\Omega)$ (supressing Ω if the context make it obvious). The set B itself will be sub-divided in various subregions $B_k \quad k > 0$ in which through each point there will pass $2k+1$ bisecting lines. (In fact the whole plane minus B would make up a natural B_0). If we talk about $2k+1$ distinct lines, those subsets B_k will be open. There boundaries will consist of points through which two of the bisections coalesce. One speaks about ramification. As we cross such a boundary the number of bisecting lines either drop or increase by an amount of two. The bisecting lines that stay distinct survive the crossing, while the two that are made to coincide do not. On the other hand going from the other direction, they will appear 'from nowhere'. The boundary points will make up a curve, consisting of all the mid-points of bisecting segments. As those are naturally parametrized by their directions we obtain a naturally parametrized curve which we will denote by $\partial B(\Omega)$ and refer to as the bisecting locus, although in general we do not expect it to coincide with the topological boundary, even if it will contain it. What kind of region B are we talking about? To find out let us make an experiment and try to plot it in a very simple case, namely the triangular region Δ made up of the three inequalities $x, y > 0$ and $x + y < 1$. But before that it would be convenient to make a digression on duality.

The space of lines

A line can be given by an equation $ax + by + c = 0$ where not all the coefficients are zero. The uninitiated reader may be puzzled by what is meant by the line $c = 0$ but that will subsequently be explained. The main point to note is that the coefficients (a, b, c) are not determined by the line, and multiple (ta, tb, tc) will do as well. One says that (a, b, c) are homogenous co-ordinates. One formal way of presenting this is to consider the space $S = R^3 \setminus \{(0,0,0)\} \sim$ where two points are identified iff they are multiples of each other in the sense above, i.e. $(a, b, c) \sim (ta, tb, tc)$ with $t \neq 0$. Geometrically we can think of this as the space of all lines in R^3 through the origin. Or in other words all the possible directions. Some of those lines can be parametrized by points on the hyperplane H given by $c = 1$ simply by considering the intersection of the line with H. Not all lines have intersection with H, those that do, form a subset isomorphic with R^2. The lines parallel to H form a set isomorphic with a circle with antipodal points identified, which is actually topologically a circle, and constitute what is called the line at infinity. One can think of those as all the directions as given by lines in R^2 through the origin, and be made up of equivalence classes of parallel lines, which can be thought of as the intersections at infinity. By the use of perspective and parallel lines converging to a vanishing point at the horizon, this notion can be given a very tangible meaning. In fact one has a natural notion of line in S, namely given by the planes through the origin. Two distinct lines will always meet at a point, as two distinct planes through the origin will always meet in a line. The set S which we have defined is called the real projective plane and denoted by $\mathbb{R}P^2$. The line $c = 0$ with co-ordinates $(0, 0, c)$ (or if you prefer normalized to $(0, 0, 1)$) corresponds to the line at infinity.

Now how should we represent the projective plane in a nice way? One way is to normalize the co-ordinates such that $a^2 + b^2 + c^2 = 1$ by choosing $t = \frac{1}{\sqrt{a^2+b^2+c^2}}$. We then get a sphere with antipodal points identified. To represent it in the plane we can consider a so called orthogonal projection which will map one hemisphere to a circle. The boundary of the circle will have anti-podal points identified. The picture we will get is the following.

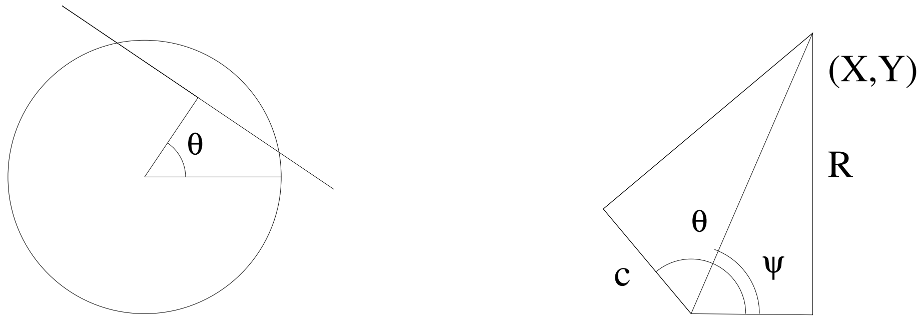




Note that in this way we get a natural metric on the space of lines, by considering them as antipodal points on a sphere. Each line in $\mathbb{R}P^2$ will naturally have length π as they can be represented by half great circles with the end points (lying at the line at infinity) identified. Now the projective plane can be thought in many ways as a compactification of the plane, simply by choosing a distinguished line and call that the line at infinity. In the picture above, the line at infinity will be the boundary circle antipodally identified.

In this context the notion of duality arises naturally. A line in R^2 gives rise to a point in our S by construction. On the other hand a point $P = (x_0, y_0)$ in R^2 gives rise to a line in S namely all the lines (a, b, c) going through the point P. The co-ordinates for those lines satisfy the linear condition $ax_0 + b_0 + c = 0$ which is the equation of a line. This sets up a principle of duality when extended to $\mathbb{R}P^2$ and its dual space $\mathbb{R}P^{2*}$, two distinct points gives rise to a unique line, two distinct lines gives rise to a unique point. It can all be elegantly expressed via the innerproduct $< x, y, z, a, b, c >= ax + by + cz$. Fixing $A = (a, b, c) \in \mathbb{R}P^{2*}$ we get a line via $0 =< X, A >= ax + by = cz$ inside $\mathbb{R}P^2$, on the other hand fixing $X \in \mathbb{R}P^2$ and consider A as variable, we get a line in $\mathbb{R}P^{2*}$.

We may also consider another way of normalizing. Any line (except the line at infinity) can be given in the form $x\cos\theta + y\sin\theta = c$ (corresponding to $(\cos\theta, \sin\theta, -c)$. We simply let θ be the angle the normal of the line makes with the x-axis



In this way we can represent the space of lines by points $(\theta, c) \in S^1 \times \mathbb{R}$ in a cylinder with antipodal points (θ, c) and $(\theta + \pi, -c)$ identified. Note that the line at infinity will not be covered by this parametrization. Now the lines through the point (X, Y) gives rise to a sinusoidal curve on the cylinder, namely by $c = R\sin(\theta - \psi)$

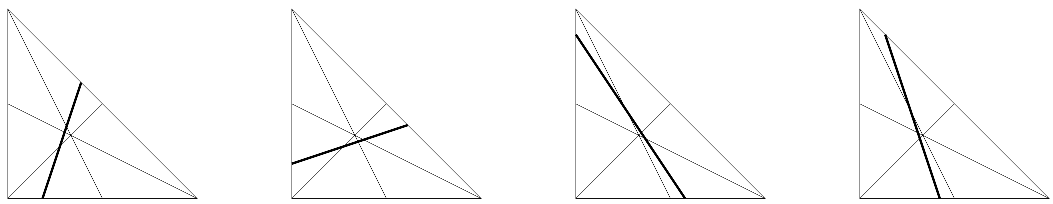
In fact this way of representing lines is given by the map projection obtained by projecting from the center of the sphere to a cylinder wrapped around the equator. We could also note in passing that if we consider directed lines, i.e. lines with an arrow, then they will form an S^2 the universal double cover of $\mathbb{R}P^2$. In the cylindrical representation the arrow will be positively oriented with respect to the normal

Finally we may note that we actually can make a canonical identification between a space and its dual given the unit circle. This allows us to associate to each point in $\mathbb{R}^2$ a line in $\mathbb{R}^2$ in the following way. If L does not go through the origin it can be represented in a unique way as $x\cos\theta + y\sin\theta = R$ with $R > 0$ (note that in our original notation $R = -c$). Similarly any point different from the origin can be written uniquely in polar form $(R\cos\theta, R\sin\theta)$ with R. Now associate to each point (θ, R) the line $(\theta, 1/R)$. The points on the unit circle will be associated to the corresponding tangent lines, while the origin will correspond to the line at infinity. The whole thing can also be run backwards, associating a line to a point. If the line intersects the circle, the corresponding tangents will intersect in a point which will set up the duality. Finally one may consider the map $(\theta, R) \mapsto (\theta, 1/R)$ which is nothing but inversion in the unit circle.

Now to return to our convex regions Ω. We may consider all the lines L which do not intersect Ω they form a subset of $\mathbb{R}^2$. The boundary of that subset will consists of lines that intersect the boundary of Ω in one point, if it is an extremal point, or along a line-segment, and subdivide the plane into two parts, one of which contains Ω. Such lines are called supporting lines, and through each point outside Ω there will be exactly two such supporting lines. While through a point inside Ω there will be no supporting lines at all. The subset $\hat{\Omega}$ of lines intersecting Ω will form a convex set in S called the dual of Ω. The bisecting lines will form a curve inside $\hat{\Omega}$ which we will refer to as the bisecting locus. The corresponding mid-points will form a curve inside Ω itself, a curve which we already have denoted by $\partial B(\Omega)$. Those two curves will be in so called duality, meaning that either of them is gioven by the other by associating their tangents. That indeed the bisecting segment is tangent to the bisecting locus will be shown later.

The Triangle

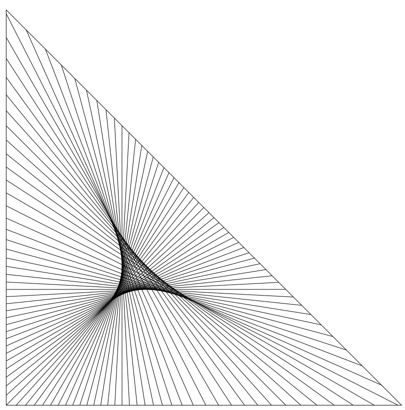
Now let us explicitly identify all the bisecting lines of Δ. It is clear that we will have four cases depending on the slope k of the line,



In each of the four cases it is straightforward to compute the area of the sliced off triangle and determine the value of c for which its area becomes $\frac{1}{4}$. By normalizing to $b \geq 0$ we get the following table (where $\alpha = \arctan \frac{1}{2}$)

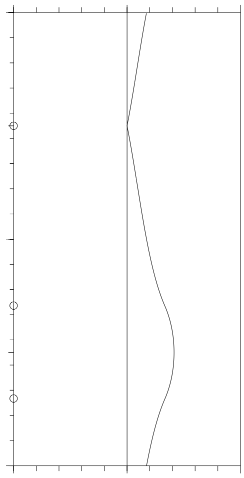
$k \geq 1$	$-c = a + \sqrt{\frac{1}{2}a(a-b)}$	$\frac{3\pi}{4} \leq \theta \leq \pi$	$\cos\theta + \frac{1}{2}\sqrt{1+\sqrt{2}\cos(2\theta+\frac{\pi}{4})}$
$-\frac{1}{2} \leq k \leq 1$	$-c = b - \sqrt{\frac{1}{2}b(b-a)}$	$\frac{\pi}{2} - \alpha \leq \theta \leq \frac{3\pi}{4}$	$\sin\theta - \frac{1}{2}\sqrt{1-\sqrt{2}\sin(2\theta+\frac{\pi}{4})}$
$-2 \leq k \leq -\frac{1}{2}$	$-c = \sqrt{\frac{1}{2}ab}$	$\alpha \leq \theta \leq \frac{\pi}{2} - \alpha$	$\frac{1}{2}\sqrt{\sin 2\theta}$
$k \leq -2$	$-c = a - \sqrt{\frac{1}{2}a(a-b)}$	$0 \leq \theta \leq \alpha$	$\cos\theta - \frac{1}{2}\sqrt{1+\sqrt{2}\cos(2\theta+\frac{\pi}{4})}$

We can now draw 'all' the bisecting lines and get the following picture.



In the middle we get an amoeba-like blob with three cusps. It is tempting to guess that the corresponding cuspidal tangents are the three medians. The following picture seems to bear it out.

Let us now fix a simple bisecting line such as the median $x = y$ and see what is going on along it.



We see that the function has a single minimum and a single maximum. Thus through each point on the segment bounded by the extremal values there will be three bisecting lines going through.

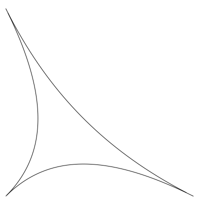
We can compute the locus of the points which are given by bisectors intersecting themselves. If the lines are parametrized by the upper part of the circle, via $C(\theta) = c(\cos\theta, \sin\theta)$ (those functions are given explicitly in table 1) then the points will be parametrized by

$$x = \begin{vmatrix} C(\theta) & \sin\theta \\ C'(\theta) & \cos\theta \end{vmatrix} \qquad y = \begin{vmatrix} \cos\theta & C(\theta) \\ -\sin\theta & C'(\theta) \end{vmatrix}$$

We will now make the necessary computations for the four cases and present below

$$\begin{array}{llll}
x = & 1 + \frac{\cos\theta + \sqrt{2}\cos(\theta+\frac{\pi}{4})}{2\sqrt{1+\sqrt{2}\cos(2\theta+\frac{\pi}{4})}} & y = & \frac{\sin\theta - \sqrt{2}\sin(\theta+\frac{\pi}{4})}{2\sqrt{1+\sqrt{2}\cos(2\theta+\frac{\pi}{4})}} \\
x = & \frac{-\cos\theta + \sqrt{2}\sin(\theta+\frac{\pi}{4})}{2\sqrt{1-\sqrt{2}\sin(2\theta+\frac{\pi}{4})}} & y = & 1 + \frac{-\sin\theta + \sqrt{2}\cos(\theta+\frac{\pi}{4})}{2\sqrt{1-\sqrt{2}\sin(2\theta+\frac{\pi}{4})}} \\
x = & \frac{\sin\theta}{2\sqrt{\sin 2\theta}} & y = & \frac{\cos\theta}{2\sqrt{\sin 2\theta}} \\
x = & 1 + \frac{-\cos\theta - \sqrt{2}\cos(\theta+\frac{\pi}{4})}{2\sqrt{1+\sqrt{2}\cos(2\theta+\frac{\pi}{4})}} & y = & \frac{-\sin\theta + \sqrt{2}\sin(\theta+\frac{\pi}{4})}{2\sqrt{1+\sqrt{2}\cos(2\theta+\frac{\pi}{4})}}
\end{array}$$

If we use those formulas to map those points they will trace out the boundary ∂B.

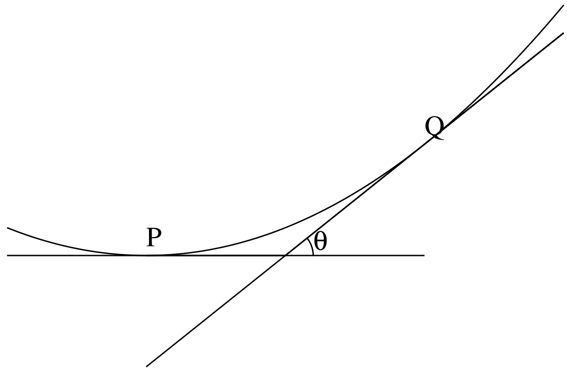


Those arcs involved are actually arcs of hyperbolas. One case is simple, namely the third, when it is easy to see that the points satisfy $xy = \frac{1}{8}$. The others we can easily work out, if not by making a linear transformation that transforms our triangle into an equilateral one placed at the its center at the origin. It is easy to see that all the objects we have defined so far will transform in the same linear way, as a bisecting line will be carried to a bisecting line under any linear transformation.

Before we will look at the general case of the bisecting locus we will make a slight digression on curvature and singularities.

Curvature and Singularities

Given a curve $C(t)$ it bends in general, meaning that the directions of the tangents vary. We talk about the total curvature between two points on it as being given by the angle made by the two tangents



Having the notion of a total curvature, we get the local by differentiation. I.e. we let the two points P, Q approach each other and divide the total curvature with the distance between and getting a limit. The angle made by the tangents is the same as being given by the normals. If the points are close the lengths of the two normals from their intersection point to the tangency points will likewise be close. In fact chose a third point R between, and consider the unique circle going through the three points P, Q, R. Its center will be given by the intersection of the two midpoints normals to the segments PR and RQ. Hence there will be an approximating circle, formed by three points coming together in analogy with two approaching points defining a tangent line. The limit of the approximating circles will be called the circle of curvature. Its center will be called the center of curvature, and its radius r the radius of curvature, while $1/r$ will be denoted the curvature. This fits well with the original definition, because the arc of an approximate circle will approximate the length of the arc between the two points. The former being $r\Delta\theta$ we are done. The locus traced by the center of curvatures is interesting and can be thought of as the dual of the curve in the dual space of lines given by the normals of the curve.

The circle of curvature is the circle who has the best fit of all the circles to the curve. In fact, as seen by the construction, a curvature circle touches the curve in three coinciding points. Would we compare the Taylor expansion of the circular arc with that of the curve they would coincide up to and including the second order term. If we use instead all quadric curves, we have instead of three parameter, five parameters, and would be able to get approximations that also included fourth order terms.

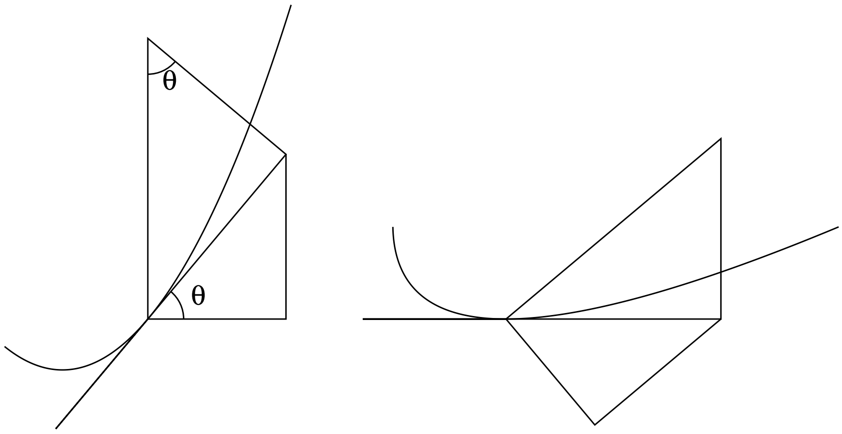
To compute the curvature is easy in the case of an horizontal tangent. We can look at the case of the parabola $y = ax^2$. The tangent line at (x_0, y_0) is given by $y - y_0 = 2ax_0(x - x_0)$ hence that of the normal will be $y - y_0 = -\frac{1}{2ax_0}(x - x_0)$ and its intersection with the y-axis $\frac{1}{2a} + ax_0^2$ and as $x_0 \to 0$ we see that the limit will be $\frac{1}{2a}$ which both gives the position of the center, and the radius of the circle. Thus the curvature is given by $2a$. An alternative is to consider a circle of radius r tangent to the x-axis at the origin. It has the equation $x^2 + (y - r)^2 = r^2$ and we can solve locally for y getting

$$y = r - \sqrt{r^2 - x^2} = r(1 - \sqrt{1 - \frac{x^2}{r^2}}) = \frac{x^2}{2r} + \dots$$

Comparing with the parabolic arc $y = ax^2$ we find that it will have the same curvature as the circle if $a = \frac{1}{2r}$, i.e. we get curvature $2a$ (and radius of curvature

$\frac{1}{2a}$). If f would be a function with the same Taylor expansion as the parabola, we would have $a = \frac{1}{2}f''(0)$ and thus curvature is given by $f''(0)$.

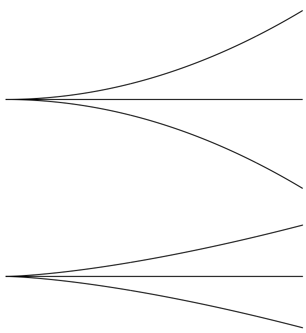
Now the curvature is intimately related to the second derivative which tells you how fast the curve bends, or rather how far it is from being a straight line. However the relation is not straightforward.



Rotating the picture to the left the new co-ordinates X, Y will be given in terms of the old as $X = x/\cos\theta$ and $Y = y\cos\theta$. If $y = \alpha x^2$ we get $Y = y\cos\theta = \alpha\frac{x^2}{\cos^2\theta}\cos^2\theta\cos\theta = \cos^3\theta\alpha X^2$. As $\tan\theta = f'(x_0)$ we get $\cos\theta = \frac{1}{\sqrt{1+(f'(x_0))^2}}$ and hence the curvature in general is given by $\frac{f''(x_0)}{(1+(f'(x_0))^2)^{\frac{3}{2}}}$.

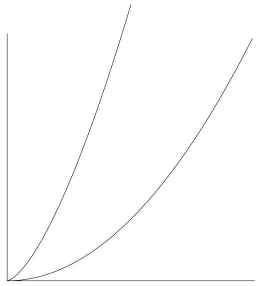
Now let us consider a piece of curve which is given in polar-coordinates by $r(\theta)$ a function which tends to zero as $\theta \to 0$. Clearly the origin will be a point on the curve, but what type of point? Assume that $\lim_{\theta\to 0} r(\theta) = A > 0$. We then get $(x'(0), y'(0)) = (A, 0)$ thus the curve is tangent to the x-axis. Writing $x \sim A\theta\cos\theta, y \sim A\theta\sin\theta$ we can for small values of θ write $y = A\theta\theta = \frac{1}{A}(A\theta)^2$ thus locally $y = \frac{1}{A}x^2 + \dots$ i.e. the curve approaches its tangent quadratically.

However if A would abruptly change sign as θ becomes negative, then the curve would back up on itself and stay on the right half plane. We will have occasion to return to this phenomenon later.



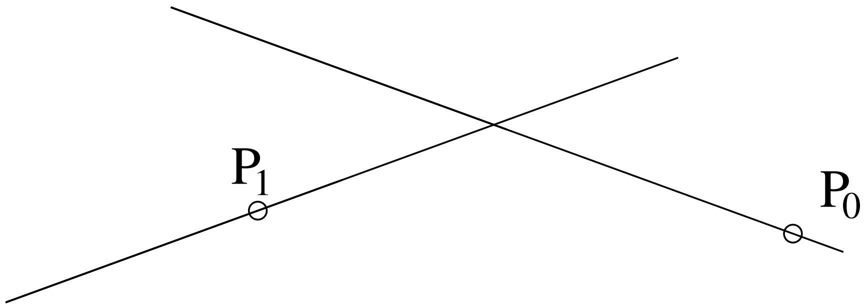
Finally what happens if $\lim_{\theta\to 0}\frac{r\theta}{\theta^2} = 0$. The same argument as before yields $y = A\theta^2\sin\theta = A\theta^3 = \frac{1}{\sqrt{A}}(A\theta^2)^{\frac{3}{2}} = \frac{1}{\sqrt{A}}x^{\frac{3}{2}}$. This is the cuspidal singularity given by the polynomial equation $Ay^2 = x^3$.

The two branches approach the tangent much slower than in the quadratic case. In fact it is a bit hard to get an intuitive approach of how it actually approaches zero.



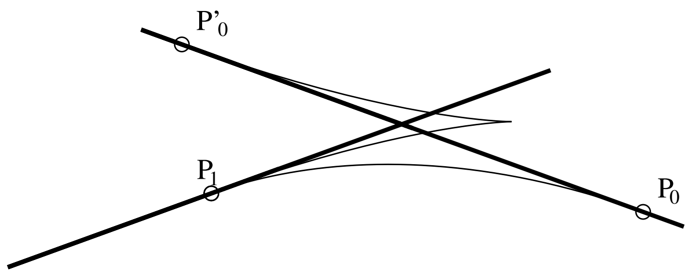
The picture to the left shows a comparison between a cuspidal approach to the tangent and a regular quadratic. While the curvature of the parabola approaches a finite non-zero value as we go to the origin, for the cuspidal curve it goes to infinity, which is however quite hard to see from a picture.

Finally we will discuss how one can interpolate between two tangents with given tangent-points. In other words to find a convex arc that joins P_0 and P_1 below, while being tangent to the corresponding lines at those points.



Write the lines as $x\cos\theta_0 + y\sin\theta_0 = c_0$ and $x\cos\theta_1 + y\sin\theta_1 = c_1$ respectively, and let us give the points as $P_0 = c_0(\cos\theta_0, \sin\theta_0) + r_0(-\sin\theta_0, \cos\theta_0)$ and $P_1 = c_1(\cos\theta_1, \sin\theta_1) + r_1(-\sin\theta_1, \cos\theta_1)$. It is then natural to consider the curve $C(\theta) = c(\theta)(\cos\theta, \sin\theta) + r(\theta)(-\sin\theta, \cos\theta)$ with the obvious boundary conditions $c(\theta_i) = c_i, r(\theta_i) = r_i$. Taking the derivative of C we get $(-(c+r')\sin\theta + (c'-r)\cos\theta, (c+r')\cos\theta + (c'-r)\sin\theta)$ from which we derive the condition $c' = r$. Now we can take any function r that interpolates between r_0 and r_1 which will determine c up to an additive constant. If r is merely linear we will not have enough freedom to fit c but if we take r quadratic it will always work out uniquely. We may also use the freedom in choosing the origin of our presentation. A shift to (A, B) will result in $\tilde{c}(\theta) = c(\theta) - A\cos\theta - B\sin\theta, \tilde{r}(\theta) = r(\theta) + A\sin\theta - B\cos\theta$. In this way by appropriate choices of A, B we may arrange to get a linear choice for $\tilde{r}$. In fact a linear choice is given by $r(\theta) = \frac{r_0+r_1}{2} + K(\theta - \frac{\theta_0+\theta_1}{2})$ from which follows that $c(\theta) = k + \frac{r_0+r_1}{2}\theta + \frac{K}{2}(\theta - \frac{\theta_0+\theta_1}{2})^2$ and hence that we need $c(\theta_1) - c(\theta_0) = \frac{r_0+r_1}{2}(\theta_1 - \theta_2)$ which can easily be effected given the freedom of (A, B).

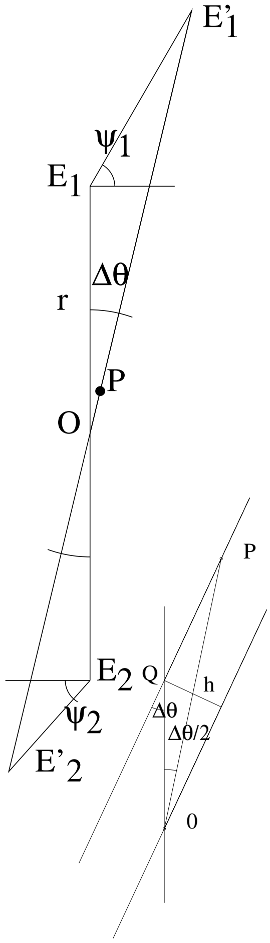
In the picture below we have done this for two choices of P_0. Note that what we are interpolating are the tangent rays, not the tangents. In the smooth case the angular change is given by the acute angle in the intersection, while in the alternative case with a cusp, we have an angular change that is greater corresponding to the obtuse angle.



We may remark that the curvature of $C(\theta)$ is easily computed to be $(c + r')$ and that the locus of centers of curvature is given by $-r'(\cos\theta, \sin\theta) + r(-\sin\theta, \cos\theta)$

The bisecting locus

We would now like to study the bisecting locus more carefully. The sequence of pictures below, suitably interpreted will give the key to many of its properties.



So let us fix one bisecting segment $L = E_1E_2$ which will make the angles ψ_1, ψ_2 respectively with the boundary $\partial\Omega$ of the region Ω and let us denote the lengths of OE_1 and OE_2 with r. Now take a neighboring bisecting segment L', making the angle $\Delta\theta$ with L. Its length up to first order will be given by the sum of the lengths of $OE_1' = r(1 + \Delta\theta \tan\psi_1)$ and $OE_2' = r(1 + \Delta\theta \tan\psi_2)$. If we take the midpoint of the segment we will not in general land in O but in P and the length of OP is $\frac{1}{2}r(\tan\psi_1 - \tan\psi_2)$. However this is not the entire story, as the areas of the triangles OE_1E_1' and OE_2E_2' are not the same, as they would be if L' was a bisecting segment. We will have to make a parallel shift of L' and in this picture to the right. The excess of the first over the second is given by the difference of the areas of the top and bottom parts respectively i.e. $\frac{1}{2}r^2(\tan\psi_1 - \tan\psi_2)(\Delta\theta)^2$. Thus if we want to shift with an amount h we will get

$$h(r(1 + \tan\psi_1) + r(1 + \tan(\psi_2) = \frac{1}{2}r^2(\tan\psi_1 - \tan\psi_2)(\Delta\theta)^2.$$

From this it follows that $h = \frac{1}{4}r(\tan\psi_1 - \tan\psi_2)(\Delta\theta)^2$. As h is of second order in $\Delta\theta$ this shift will not affect the length of L' to the first order. The length k of QP will be given by $k \sin\Delta\theta = h$ thus to be given by $k = \frac{1}{4}r(\tan\psi_1 - \tan\psi_2)\Delta\theta$. The point P on L' will hence be moved that amount further up (in our case). Finally if we draw the segment PO this will have length $\frac{1}{2}r(\tan\psi_1 - \tan\psi_2)\Delta\theta$ and make the angle $\frac{\Delta\theta}{2}$ with L.

From this we can draw a number of conclusions.

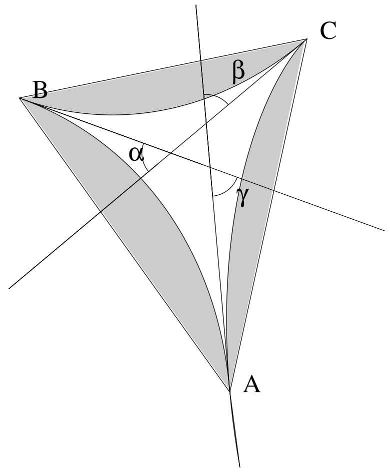
Conclusion *If $\psi_1 \neq \psi_2$ then O is a smooth point of ∂B and the bisecting line L is tangent to ∂B at O. Furthermore it lies on the concave side of the curve. The speed of the natural parametrization of ∂B is given by $\frac{1}{2}r|(\tan\psi_1 - \tan\psi_2)|$ Finally the curvature at the point O is given by $\frac{2}{r}(\tan\psi_1 - \tan\psi_2)^{-1}$.*

PROOF: By construction as P approaches O the angle it makes with the line L goes to zero, this line will hence be the tangent. As $\Delta\theta$ switches sign the point P will still stay at the same side. We note also that the point Q on the tangent line L sits inside B because the latter is made up of such intersection points. The formula for the speed is immediate. As for the curvature we refer to a previous section for the following approach.

If we intersect the parabola with a line of slope $\tan(\Delta\theta/2)$ through the origin the distance to the other intersection point will be of the order $\frac{1}{2a}\Delta$. Comparing with what we have above we conclude that the radius of curvature will be $\frac{1}{2}r(\tan\psi_1 - \tan\psi_2)$ hence the formula.

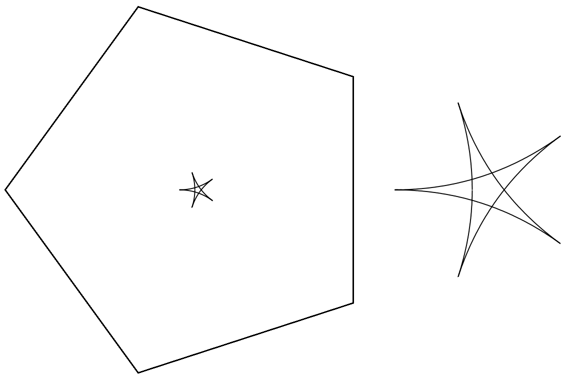
We also note that if the end point belongs to a corner, the length of the radius vector will so to speak switch signs, and thus we will get two tangent branches. It is easy to work out the conditions for this, using the methods above. It has to do with the acuteness of the corner, and how slope at the opposing point. Precise statements are safely left to the interested reader.

We note that the curve ∂B cannot enclose a convex region B as in that case for each direction θ there will be two tangents, but each direction only contains one bisector. In fact ∂B will consist of a number of concave arcs joined together by singularities, and have the property that no two tangents are parallel. We cannot build an enclosing curve with just two concave arcs, at least three is needed as in our example of the triangle. But more than three will not work either if we insist on a simple curve, as the following picture explains.



One should think of the points AB etc as being joined by a concave arc tangent at the appropriate lines. The total curvature will then be given by γ etc. The total contribution $\alpha + \beta + \gamma$ is then seen as π the maximum allowed. If a concave arc is split up into two parts joined by a cusplike singularity, the total curvature will increase.

A more complicated curve is given by the following example of a regular pentagon

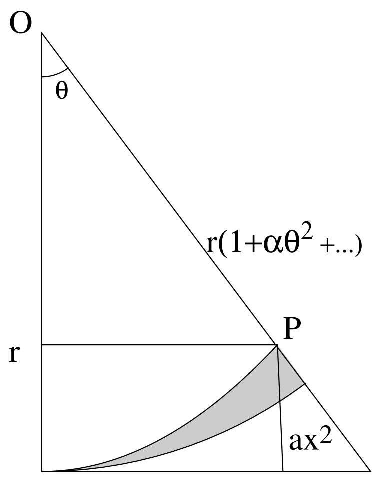


Finally we note that bisecting segments are not the only chords that can be associated to a convex region. As we have already noted, for each direction there are two supporting lines. They touch the boundary at two points that can be joined. That segment may or may not be a bisecting one but if it is then it corresponds to a critical one, i.e. when the direction of the boundary coincide at opposite points, and conversely every critical bisecting segment occurs in this way. Furthermore between two supporting lines there will be a bisecting one, as well as one which lies half-way between, and which will sometimes also be a bisecting segment. However, if we trace the self-intersection points there will sometimes be jumps, when the boundary of the region has segments and corners.

Now let us consider the singularities that occur on the curves.

Singularities

We assume that the bisecting segment is vertical and orthogonal to the boundary $\partial\Omega$ at its endpoints, and those arcs are given locally as $r(1 - ax^2 + \dots)$ and $r(1 - bx^2 + \dots)$. Close bisectors making a small angle $\Delta\theta$ will have radii of lengths $r(1 + \alpha\Delta\theta^2)$ and $r(1 + \beta\Delta\theta^2)$ respectively. How do we compute those lengths? For that purpose we look at the picture below.



Making the 'Ansatz' above we get

$x = r(1 + \alpha\Delta\theta^2 + \dots) \sin\Delta\theta$

and the equation

$\cos\Delta\theta r(1 + \alpha\Delta\theta^2 + \dots..) = r - a(\sin\Delta\theta r(1 + \alpha\Delta\theta^2 + \dots))^2$ or $(1 - \frac{\Delta\theta^2}{2} + \dots)(1 + \alpha\Delta\theta^2 + \dots) = 1 - a\Delta\theta^2 r(1 + \dots)$

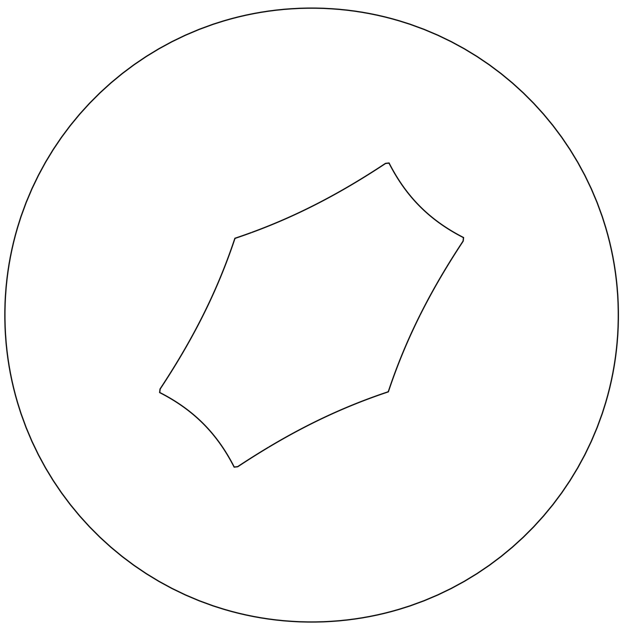
from which follows $(\alpha - \frac{1}{2}) = -ar$, thus $\alpha = 0$ is equivalent with $a = \frac{1}{2r}$ i.e. the circle of radius r is the circle of curvature to the parabolic arc.

In order to compute the shift h as in the previous exercise, we need to be able to compute areas of the shaded region (which in this case is the defect to make up a circular segment). Those are given by a simple integration $\int_0^{\delta\theta} r^2\alpha\theta^2 d\theta = \frac{1}{3}r^2\alpha\Delta\theta^3$.

As before we need to have $2rh = \frac{1}{3}r^2\alpha\Delta\theta^3 - \frac{1}{3}r^2\alpha\Delta\theta^3$ the shift will be given by $h/\sin\Delta\theta = \frac{1}{6}r^2(\alpha - \beta)\Delta\theta^2)$ while the shift from the extended lengths will be given by $\frac{1}{2}r(\alpha - \beta)\Delta\theta^2$. The important thing is that the length of the radius vector of angle $\Delta\theta$ will no be quadratic and not linear in the angle. This corresponds to an approach given by $r(\theta) \sim \theta^2$ in polar co-ordinates, thus $x \sim \theta^2$ and $y \sim \theta^3$. In other words a cusp $y^2 = x^3$. Note however that in the case of our triangle, the singularties will be so called tacnodal, due to the corners of $\partial\Omega$ and correspond to two tangent branches.

The Symmetrizer

We may translate the bisecting segments such that their midpoints land on the origin. What we get is a star-shaped region and which is symmetric with respect to reflection in the center. Let us give it a name -the symmetrizer $S(\Omega)$ of Ω. In the case of our triangle we get the following non-convex region



What is the area of $S(\Omega)$? There is a natural map Φ from $S^* = S(\Omega) \setminus \{(0,0)\}$ to $\Omega \setminus \partial B$ which 'blows up' the origin to the curve ∂B. Any point $p \in S^*$ lies on a unique ray through the origin, which corresponds to a unique point $P \in \partial B$ which defines a translation T which carries $(0,0)$ to P. Then define $\Phi(p) = T(p)$. Intuitively this map is area-preserving. Locally it can be well-approximated by so called 'shearing maps'. Those are maps that for some choice of co-ordinates can be written under the form $(x, y) \to (x, y + h(x, y))$ for some function h which need not even be continuous, let alone differentiable, only measurable n general. What that map does is to translate small squares without overlapping or tearing. In the case h is differentiable it is straightforward to compute its Jacobian and check that it is identically equal to one. In our case we can also write down Φ explicitly by using a parametrization ϕ_0, ϕ_1 of ∂B by writing down

$$(x, y) \mapsto (x + \phi_0(\arctan \frac{y}{x}), y + \phi_1(\arctan \frac{y}{x}))$$

The Jacobian will be given by the determinant

$$\begin{vmatrix} 1 + \phi_0'(\arctan \frac{y}{x})\frac{-y}{x^2+y^2} & \phi_1'(\arctan \frac{y}{x})\frac{x}{x^2+y^2} \\ \phi_0'(\arctan \frac{y}{x})\frac{-y}{x^2+y^2} & 1 + \phi_1'(\arctan \frac{y}{x})\frac{x}{x^2+y^2} \end{vmatrix} = 1 + \frac{-y\phi_0' + x\phi_1'}{x^2 + y^2} = 1$$

as the bisecting curve is tangent to the defining line.

Note that the map defined has a strange property as it approaches the origin. Although the quotient $|\Phi(p) - \Phi(q)|/|p - q|$ goes to infinity as p, q approaches the origin, the map is nevertheless area-preserving.

Now the following should be obvious, where μ denotes area.

Proposition $\mu(S(\Omega)) = \mu(\Omega) + 2\sum_k k\mu(B_k(\Omega))$

PROOF: We simply remark that on the inverse image $\Phi^{-1}(B_k)$ the map is $2k+1 : 1$.