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Dilations and the arbelos

Hiroshi Okumura

251 Moo 15 Ban Kesorn, Tambol Sila
Amphur Muang Khonkaen 40000, Thailand
hiroshiokmr@gmail.com

1 Introduction

For a point O on a segment AB with $|OA| = 2a$ and $|OB| = 2b$, let α, β and γ be semicircles with diameters AO, BO and AB respectively erected on the same side. The area surrounded by the three semicircles is called an arbelos. The radical axis of α and β divides the arbelos into two curvilinear triangles with congruent incircles, which are called the twin circles of Archimedes (see Figure 1). Leon Bankoff found that the circle orthogonal to α, β and the incircle of the arbelos is congruent to the twin circles [1] (see Figure 2). Circles congruent to the twin circles are said to be Archimedean. The common radius of the Archimedean circles is $ab/(a + b)$, which is denoted by r_{A}. In this article we generalize the twin circles, also we get some other new infinite Archimedean circles whose centers lie on a conic section by using dilations with center O.

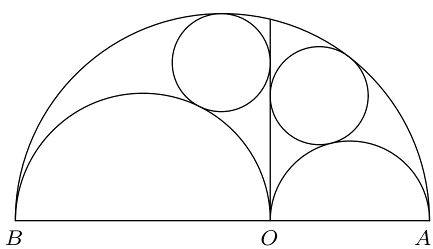

Figure 1.

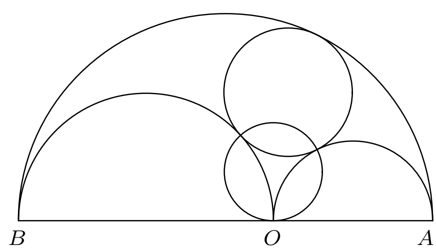

Figure 2.

We use a rectangular coordinate system with origin O such that the coordinates of A and B are $(2a, 0)$ and $(-2b, 0)$ respectively. The radical axis of α and β is denoted by $\mathcal{L}$. Let σ be a dilation with center O and scale factor $k > 0$. The image of a point P by σ is denoted by P^σ. For two points P and Q on the line AB, (PQ) denotes the semicircle with diameter PQ, where all the semicircles are constructed in the region $y > 0$.

2 The twin circles of Archimedes

In this section, we generalize the twin circles of Archimedes. Floor van Lamoen has found that for a dilation τ with center A, the circle touching the semicircles (AO^τ) externally (AB^τ) internally and the line $\mathcal{L}$ from the side opposite to B is Archimedean [2]. A similar property also holds for the dilation with center O (see Figure 3).

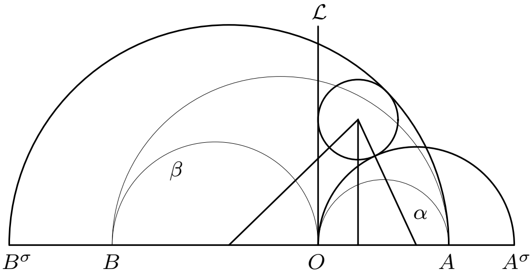

Figure 3: $k = 1.5$

Theorem 1. *The circle touching the semicircles (OA^σ) externally (AB^σ) internally and the line $\mathcal{L}$ from the side opposite to B is Archimedean.*

Proof. Let x be the radius of the touching circle. By the Pythagorean theorem

$$((a+kb)-x)^2 - ((a-kb)-x)^2 = (ka+x)^2 - (ka-x)^2.$$

Solving the equation we get $x = r_{\mathrm{A}}$. □

One of the twin circles touching the semicircle α is obtained when σ is the identity. By exchanging the roles of the points A and B we get one more Archimedean circle.

Theorem 2. *The circle touching the semicircles $(OA^{\sigma^{-1}})$ externally and $(A^\sigma B^{\sigma^{-1}})$ internally and the line $\mathcal{L}$ from the side opposite to B has radius kr_{A}. The point of tangency of the touching circle and $\mathcal{L}$ coincides with the point of tangency of one of the twin circles touching α and $\mathcal{L}$.*

Proof. The first part is proved similarly to Theorem 1 (see Figure 4). The second part follows from the fact: the segment length of the common external tangent of two externally touching circles with radii p and q are $2\sqrt{pq}$. □

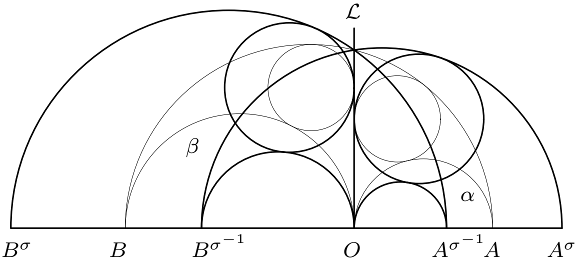

Figure 4. $k = 1.5$

Exchanging the roles of the points A and B, we get one more circles of radius kr_{A}. And the twin circles are obtained when σ is the identity. The semicircle $(A^{\sigma}B^{\sigma^{-1}})$ belongs to the pencil of circles determined by the semicircle (AB) and the line $\mathcal{L}$.

3 Two infinite sets of Archimedean circles

From now on, we include the case in which σ has a negative scale factor k, i.e., if $k < 0$ then $\overrightarrow{OP^{\sigma}} = -|k|\overrightarrow{OP}$ for any point P. Let $\alpha(k) = (OA^{\sigma})$ and $\beta(k) = (OB^{\sigma})$[0]. Also we denote the line $x = 2kr_{\text{A}}$ by $\mathcal{P}_k$. Hence $\mathcal{P}_0 = \mathcal{L}$, and $\mathcal{P}_1$ touches the Archimedean circle touching α, γ and $\mathcal{P}_0$. The next theorem is also proved similarly to Theorem 1 (see Figures 5 and 6).

Theorem 3. *Let k be a real number.*
(i) *If $0 < k$, then the circle touching $\alpha(k)$ externally, $\alpha(k+1)$ internally and $\mathcal{P}_k$ from the side opposite to the point O is Archimedean.*
(ii) *If $-1 \leq k < 0$, then the circle touching both $\alpha(k)$ and $\alpha(k+1)$ externally and $\mathcal{P}_{-k}$ from the side opposite to the point A is Archimedean.*

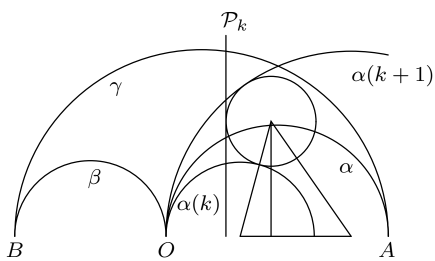

Figure 5. $0 < k$

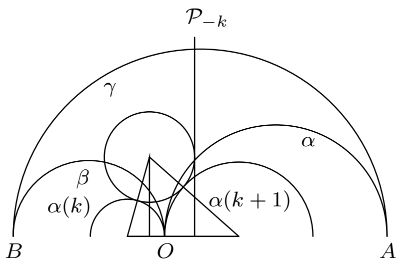

Figure 6. $-1 < k < 0$

From (i) in the theorem we get an infinite set consisting of Archimedean circles touching $\alpha(k)$, $\alpha(k+1)$ and $\mathcal{P}_k$ for some positive real number k. The line $\mathcal{P}_1$ passes through the point of intersection of $\alpha(2)$ and γ. This can be proved by using elementary properties of chords with the Pythagorean theorem. In [3] a more general aspect is considered. Therefore the circle touching α externally $\alpha(2)$ internally and the perpendicular to AB through the point of intersection of γ and $\alpha(2)$ from the side opposite to O is Archimedean from the case $k = 1$ (see Figure 7).

From (ii) we also get an infinite set consisting of Archimedean circles touching $\alpha(k)$, $\alpha(k+1)$ and $\mathcal{P}_{-k}$ for a real number k satisfying $-1 \leq k < 0$.

4 Conic sections

Let (x, y) be the center of the Archimedean circle obtained by (i) in Theorem 3. Then $x = (2k+1)r_{\text{A}}$. While by the Pythagorean theorem, $y^2+(x-ka)^2 = (r_{\text{A}}+ka)^2$.

[0]Those notations are slightly changed from the ones in [3] and [4].

Eliminating k from the two equations and rearranging, we get

$$\frac{x^2}{r_{\mathrm{A}}^2} - \frac{y^2}{r_{\mathrm{A}}^2 a/b} = 1.$$

Therefore (x, y) lies on a part of the hyperbola lying in the quadrant I with focal points $(\pm\sqrt{ar_{\mathrm{A}}}, 0)$ and asymptote

$$y = \sqrt{\frac{a}{b}}x.$$

Conversely, any point on this curve can be obtained as a center of an Archimedean circle determined in (i). The asymptote, denoted by the dotted line in Figure 7, passes through the point of intersection of $\alpha(k)$ and $\mathcal{P}_k$.

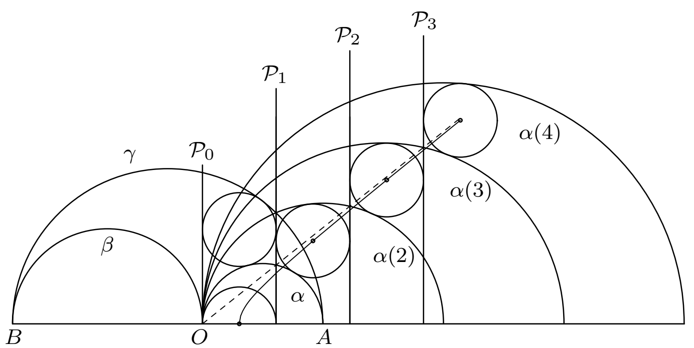

Figure 7.

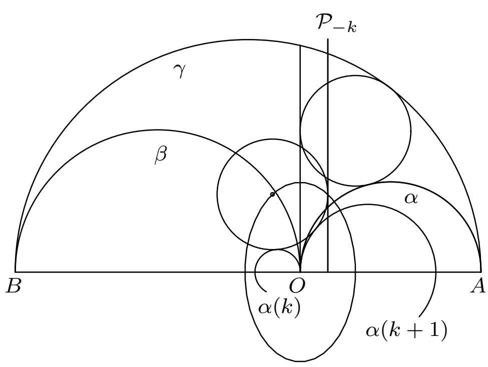

Figure 8. $k = -0.25$

Let (x, y) be the center of the Archimedean circle determined in (ii), then we can also get

$$\frac{x^2}{r_{\mathrm{A}}^2} + \frac{y^2}{r_{\mathrm{A}}^2 (a + 2b)/b} = 1.$$

Therefore (x, y) lies on a part of the ellipse lying in the region $y > 0$ together with the point $(r_A, 0)$ with minor axis $2r_{\mathrm{A}}$ and major axis $2r_{\mathrm{A}}\sqrt{(a + 2b)/b}$ and focal points $\left(0, \pm\sqrt{ar_{\mathrm{A}}}\right)$ (see Figure 8). Conversely, any point on this curve can be obtained as a center of an Archimedean circle determined in (ii). The Archimedean circle touching α, γ and $\mathcal{P}_0$, touches $\mathcal{P}_0$ at the point $(0, 2\sqrt{ar_{\mathrm{A}}})$. Therefore the focal points are obtained as the midpoint of the line segment joining O and the tangent point and its reflection in the line AB.

Both the conic sections are expressed as follows:

$$\frac{x^2}{r_{\mathrm{A}}^2} \mp \frac{y^2}{r_{\mathrm{A}}^2(a+b\mp b)/b} = 1.$$

Acknowledgments
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