Are those Archimedean triplet circles really triplets?

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1 Introduction

For a point O on the segment AB with $|AO| = 2a$ and $|BO| = 2b$, let α, β and γ be the semicircles with diameters AO, BO and AB respectively erected on the same side. The area surrounded by the three semicircles is called an arbelos (see Figure 1). The radical axis of the two inner semicircles divides the arbelos into two curvilinear triangles with congruent incircles. The circles were studied by Archimedes, and are called the twin circles of Archimedes. Circles congruent to the twin circles are called Archimedean circles, whose radii is $ab/(a+b)$.

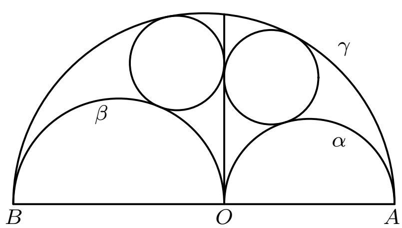

Figure 1.

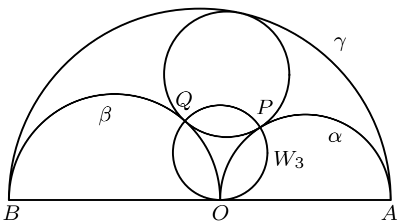

Figure 2.

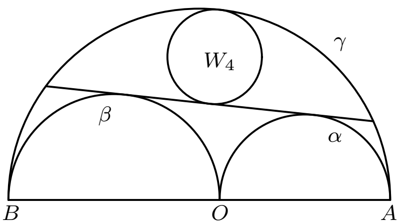

Figure 3.

More than two thousand years after, Leon Bankoff found another Archimedean circle. If the incircle of the arbelos touches the semicircles α, β at points P and Q, the circle W_3 passing through the three points P, Q and O is Archimedean (see

Figure 2) [1]. With this circle he asserted that the twin circles are two members of the triplet. Later he found one more Archimedean circle W_4, which is the maximal circle touching the external common tangent of α and β and the circular arc of the semicircle γ cut by the tangent internally [2] (see Figure 3). Since many kinds of Archimedean circles have been found today, there should be some reason to designate these circles as a triplet. In this sense, it seems irrelevant that the twin circles and the circle W_3 should be regarded as a triplet. In this article we show that the circle W_4 forms a real triplet with the twin circles. Also we show that there are infinite triplet circles (infinite pairs of three congruent circles) in the arbelos.

2 The Archimedean triplet circles

We use the following lemma in the old Japanese geometry [5].

Lemma. *A circle C with radius r is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t. If two externally touching circles C_1 and C_2 with radii r_1 and r_2 also touch the chord t and the other arc of the circle C, then h, r, r_1 and r_2 are related as*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}.$$

Proof. We reproduce the proof in [3] for the convenience of the reader (see Figure 4). The centers of C_1 and C_2 can be on the opposite sides of the normal dropped on t from the center of C or on the same side of this normal. From the right triangles formed by the centers of C and C_i $(i = 1, 2)$, the line parallel to t through the center of C, and the normal dropped on t from the centers of C_i,

$$\left|\sqrt{(r-r_1)^2-(h+r_1-r)^2} \pm \sqrt{(r-r_2)^2-(h+r_2-r)^2}\right| = 2\sqrt{r_1 r_2},$$

where we used the fact that the segment length of the common external tangent of C_1 and C_2 between the tangency points is equal to $2\sqrt{r_1 r_2}$. The formula of the lemma follows from this equation. □

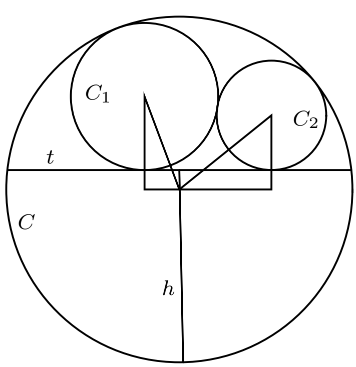

Figure 4.

If C_3 is the circle with radius $r_3 = h/2$ touching the chord t and the circle C in the lemma (see Figure 5), r, r_1, r_2 and r_3 are related as

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 2\sqrt{\frac{r}{r_1 r_2 r_3}}. \tag{1}$$

Since (1) is symmetric in r_1, r_2 and r_3, it also holds when we change the roles of the circles C_1, C_2 and C_3 as in Figures 6 and 7.

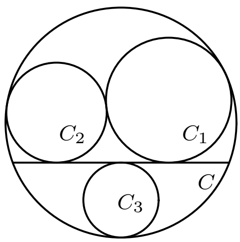

Figure 5.

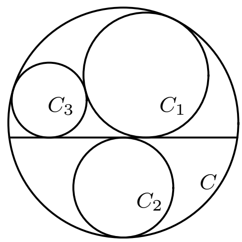

Figure 6.

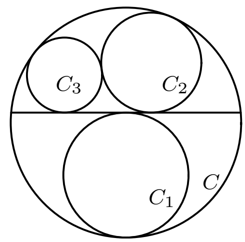

Figure 7.

Therefore we get:

Theorem. *Let C and C_i $(i = 1, 2, 3)$ be circles of radii r and r_i respectively and let C be divided by a chord t into two arcs. If two of C_1, C_2, C_3 touch externally and also touch t and one of the arcs, and the remaining is the maximal circle touching t and the remaining arc, then r, r_1, r_2 and r_3 are related as (1).*

If we regard (1) as a quadratic equation for $1/\sqrt{r_3}$, it has two positive solutions. Therefore solving (1) for r_3, we always get two positive solutions, one of which is the radius of C_3. The other is equal to the radius of the circle different from C_3 but satisfying the same condition satisfied by C_3 as denoted by the hatched lines in Figures 8, 9 and 10. We call the circle the conjugate of C_3 with respect to C_1 and C_2.

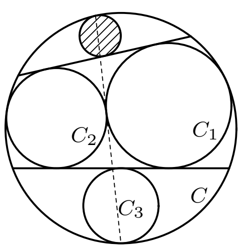

Figure 8.

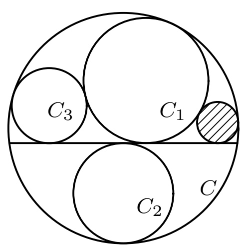

Figure 9.

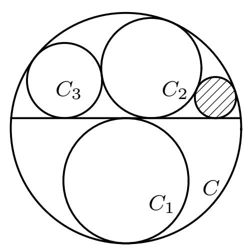

Figure 10.

Let us assume $r = a + b$, $r_1 = a$ and $r_2 = b$ in Theorem. In this case, the centers of the circles C, C_1 and C_2 are collinear. Therefore the figure consisting of the three circles is symmetric in this line. This implies that the conjugate of C_3 with respect to C_1 and C_2 is congruent to C_3, i.e., (1) has only one double root (see Figures 11, 12, 13). This implies that the twin circles and W_4 are congruent. Actually (1) gives $r_3 = ab/(a+b)$ in this case. As we have just seen, the congruence of the three circles is obtained from the same equation at the same time. Hence we may say that the three circles form a real triplet.

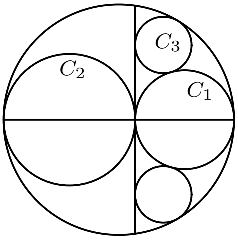

Figure 11.

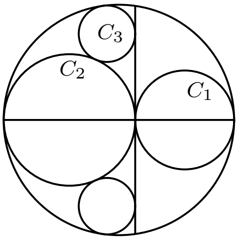

Figure 12.

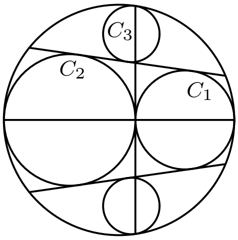

Figure 13.

If two circles C_1 and C_2 are fixed, the product of the radii of the circle C_3 and its conjugate conjugate with respect to C_1 and C_2 is constant for a given circle C by (1). It equals $(ab/(a+b))^2$ if a and b are the radii of C_1 and C_2. The same assertion can also be found in Japanese geometry in the case of Figure 8 [4].

Let t be the line lying along the chord of C touching C_3 in Figure 8. Let us consider the inversion in the circle with center at the point of tangency of C and C_3 passing through the points of intersection of C and t. By this inversion t and C are interchanged, while C_1 and C_2 remain unchanged. Therefore they are orthogonal to the inversion circle. Hence the internal common tangent of the circles C_1 and C_2 passes through the center of the inversion.

3 Infinite triplets

In this section we show that there are infinite pairs of triplet circles in the arbelos. We now observe that α, β and γ are not semicircles but circles. Let $\delta_0^1 = \delta_0^2 = \delta_0^3 = \beta$. Let δ_1^1 and δ_1^2 be the twin circles of Archimedes touching α and β respectively. To avoid overlapping figures, let δ_1^3 be the reflected image of the circles W_4 in the line AB (see Figure 14). For $i = 1, 2, 3$, let us assume that the circles δ_0^i, δ_1^i, δ_2^i, $\cdots$, δ_k^i are defined ($k \geq 1$), where δ_j^1, δ_j^2, δ_j^3 are congruent for $j = 0, 1, 2, \cdots, k$. Then δ_{k+1} is the conjugate of δ_{k-1} with respect to α and δ_k. Now the circles δ_0^i, δ_1^i, δ_2^i, $\cdots$, δ_k^i, $\cdots$ are defined.

By the definition, δ_k^1, δ_k^2, δ_k^3 are congruent for any non-negative integer k. Also from the definition, (i) If k is even, δ_k^1 is the maximal circle touching a chord t of γ and the arc of γ cut by t, i.e., it touches t from the side opposite to α. Hence it does not touch α if $k \neq 0$, since a chord of γ touches α at its midpoint if and only if it lies along the radical axis of α and β. While δ_k^1 touches α if k is odd. (ii) δ_1^2, δ_2^2, $\cdots$, δ_k^2, $\cdots$ are chain of circles touching γ and the radical axis of α and β from the side opposite to α. (iii) δ_k^3 touches α if k is even and does not touch α if k is odd. The three statements imply that δ_k^1, δ_k^2, δ_k^3 are different for any natural number k. Therefore we get infinite pairs of three congruent but different circles.

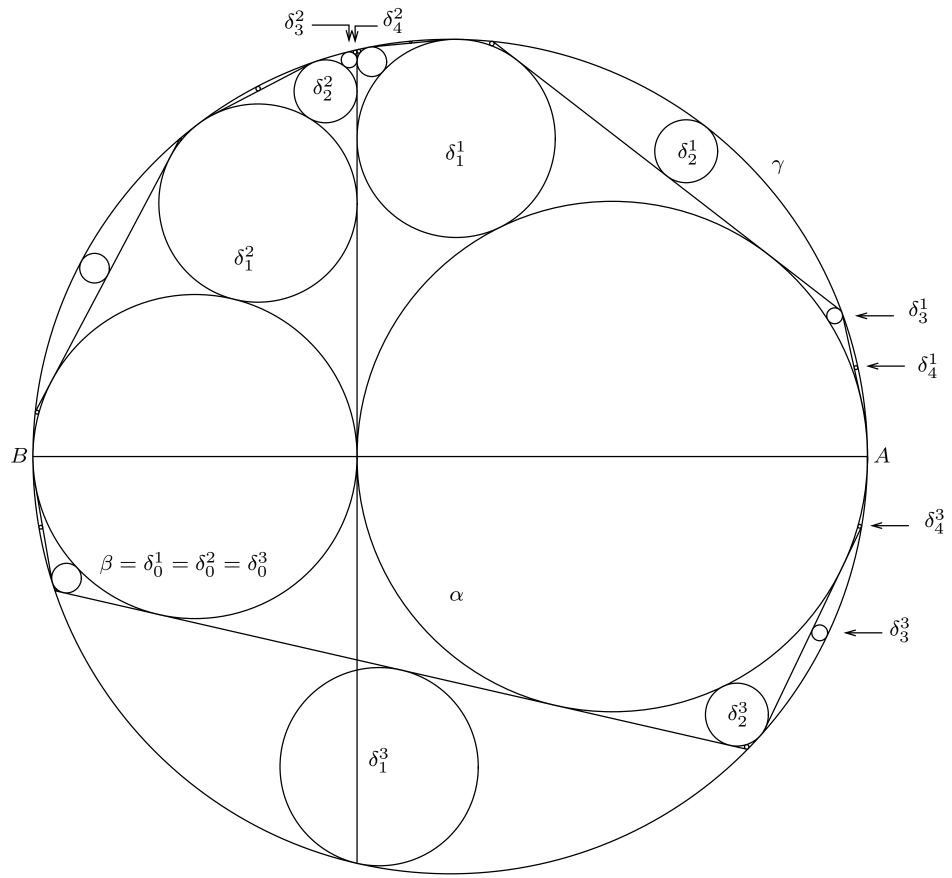

Figure 14.

We now consider each of the triplets as a set, which is denoted by a point on the two arcs in Figure 15, where $D_n = \{\delta_n^1,\ \delta_n^2,\ \delta_n^3\}$. Three points forming vertices of a triangle describe that they consist of circles satisfying the hypothesis of the theorem with the circle γ. For two triangles with one side in common in the figure (e.g. $D_1D_2P_1$ and $D_1D_2\{\alpha\}$), each of the two opposite vertices is the set consisting of conjugate of each of the circles belonging to the other vertex with respect to circles belonging to the common vertices. The figure shows that the infinte triplets expressed by D_1, D_2, D_3, $\cdots$ do not exhaust all the possible triplets. For example, triplets expressed by P_1, P_2, P_3, $\cdots$ do not appear in the triplets expressed by D_1, D_2, D_3, $\cdots$.

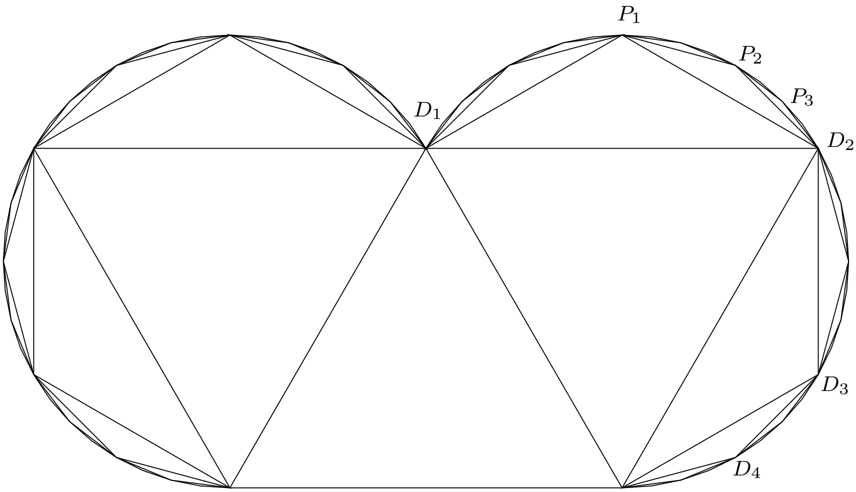

Figure 15.

Acknowledgments
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