A note on series of positive numbers

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1 Introduction

The study of series appears everywhere in Analysis. And the first issue is to know whether the series is convergent or not. Most of the times we need to appeal to absolute convergence and, in this way, we end up by trying to understand series of positive numbers. There are several criteria to decide if a series of positive numbers is convergent or not, however most of them seems to have two similar characteristics: first, they come, in one way or another, from the Comparison Principle. In certain sense, one could think that this fact is limiting our study of convergence of series of positive numbers. Second, none of them gives equivalent conditions. For example D'Alambert's, Cauchy's or Raabe's criteria fail when the corresponding limit is 1.

Here we present another criterion which gives an equivalent condition for the convergence of a series of positive numbers, which in fact does not come from the Comparison Principle. This criterion came to me when I was trying to prove to the students that the dual of L^2 is L^2 in an elementary way, now Corollay 7 below. By gravitation law, I came needing to prove Theorem 2. Searching and asking to experts, one could conclude that the result is not entirely known to the experts, even though seems a nice an useful result as one can see for the applications included here. Even though finally some partial references, included in the bibliography, appeared in the process it was only the referee who mentioned to me the excellent book by de la Vallée Poussin (7). In page 432 of this book there is a result presented as Exercise 3, that contains basically all the results presented here, as well as the results in (1) and (3), as particular cases. In the note I include both the Exercise of de la Vallée Poussin's book, and also the theorem that appeared naturally to me, together with a short, selfcontained proof, and also several applications that enlight the power of this criterion, which could be considered as part of the standard theory of series of positive numbers.

Theorem 1 *(De la Vallée Poussin) Let $a_n \geq 0$ and $\sum a_n$ divergent. Suppose $f(x)$ is decreasing and $\lim_{x\to\infty} f(x) = 0$. Let $S_N = \sum_{k\leq N} a_k$ and $F(x) = \int_0^x f(t)dt$. Consider*

$$S_{f,1} = \sum f(S_n)a_n \qquad S_{f,2} = \sum f(S_{n-1})a_n.$$

Then,

1. *If $F(x)$ is bounded then $S_{f,1}$ is convergent.*
2. *If $F(x)$ is unbounded then $S_{f,2}$ is divergent.*
3. *If a_n is bounded, then $S_{f,1}$ and $S_{f,2}$ are both convergent or both divergent at the same time.*

Theorem 2 *Let $a_k > 0$ for $k \geq 0$, and $S_N = \sum_{k\leq N} a_k$. Then,*

1. *$\sum a_k$ converges if and only if $\sum \frac{a_k}{S_k}$ does.*
2. *$\sum \frac{a_k}{S_k(\log(S_k+1))^2}$ is always convergent.*

Remark. Note that in Theorem 1 the monotony condition on $f(x)$ already guarantees the existence of $F(x)$. Also, the "difficult" implication of part (1) of the Theorem 2 is already proved in a paper by Abel in 1828, (1). Dini in 1867 in (3) improved his result and obtained the convergence of the series $\sum_{n\neq N} \frac{a_n}{S_n^\alpha}$ for $\alpha > 1$. However, their proofs, more involved than the showed here, lose enough so it is not achieved the second part of Theorem 2. The result when $\sum a_n$ is convergent, is not a consequence of Theorem 1. We give an example below.

2 Proofs.

Before proving the theorem, we should add some remarks. First let us say that Theorem 2 is in fact a criterion for series of positive numbers. Indeed, otherwise it could happen that $\frac{a_k}{S_k}$ is not well defined for infinitely many k. But even if this is not the case, one can not ensure the result. Let us for example consider $a_1 = 1$ and $a_k = 3(-1/2)^{k-1}$ for any $k \geq 2$. Then $\sum a_k$ is convergent by Leibniz's criterion. However, for any $k \geq 2$, $S_k = \sum_{j=1}^k a_j = \frac{1}{3}a_k$, and so $\sum \frac{a_k}{S_k}$ is divergent. Also, we should note that the second part of the theorem would not remain true by removing 1 from the logarithm. To see this, consider $a_k = \frac{1}{k(k+1)}$. Then, $S_k = 1 - \frac{1}{k+1}$, $|\log S_k| < \frac{2}{k+1}$, and so $\sum \frac{a_k}{S_k(\log(S_k))^2} > \frac{1}{4}\sum \frac{1}{1-\frac{1}{k+1}}$, is a divergent series. Notice that in this case S_k is convergent. Clearly this is the only case in which adding 1 to the argument of the logarithm is an important matter.

2.1 Proof of the Theorem 1.

To prove Parts (*1*) and (*2*), we note that, since f is decreasing,

$$f(S_n)a_n \leq \int_{S_{n-1}}^{S_n} f(t)dt \leq f(S_{n-1})a_n$$

and so

$$\sum_{n\le N} f(S_n)a_n \le \int_0^{S_N} f(t)dt \le \sum_{n\le N} f(S_{n-1})a_n.$$

The result follows by taking limits when $N\to\infty$. To prove Part (*3*), observe that

$$0\le \sum_{n\le N}\left(f(S_{n-1})-f(S_n)\right)a_n \le K\sum_{n\le N}\left(f(S_{n-1})-f(S_n)\right) = K(f(0)-f(S_N)),$$

whenever $a_n \le K$, by the monotonicity of f. Again, taking limits we find that $0\le S_{f,2}-S_{f,1}\le Kf(0)$ and the result follows.

2.2 Proof of the Theorem 2.

If $S=\sum a_k$ is convergent, both results in the theorem are trivial. Indeed, note that in this case $S_k > S-\varepsilon$ for any $\varepsilon>0$ and k sufficiently large depending on ε. Then, we assume S_N defines a divergent series and hence, by dropping the first terms we can assume without loss of generality that $S_1>1$.

• Part (*1*). We have to prove that $\sum \frac{a_k}{S_k}$ is divergent. This is a particular case of Theorem 1 with $f(x)=\frac{1}{x}$. We include here the proof I gave to the students.

Let us start by observing that

$$\begin{aligned}\log(S_K) &= \log\left(\prod_{k=1}^{K}\frac{S_k}{S_{k-1}}\right) = \sum_{k=1}^{K}\log\left(\frac{S_k}{S_{k-1}}\right)\\ &= -\sum_{k=1}^{K}\log\left(\frac{S_{k-1}}{S_k}\right) = -\sum_{k=1}^{K}\log\left(1-\left(1-\frac{S_{k-1}}{S_k}\right)\right) = -\sum_{k=1}^{K}\log\left(1-\frac{a_k}{S_k}\right)\end{aligned} \quad (1)$$

If $a_k\ne o(S_k)$, the result is trivial. Hence, we assume $\lim_{k\to\infty}\frac{a_k}{S_k}=0$, and so, for $k>K_0$, $0<\frac{a_k}{S_k}<\frac{1}{2}$. Then, the inequality

$$x < -\log(1-x) < 2x \quad (2)$$

valid for any $0<x<\frac{1}{2}$, gives us for any $K>K_0$ in (1),

$$\log(S_K) < -\sum_{k=1}^{K_0}\log\left(1-\frac{a_k}{S_k}\right) + 2\sum_{j=K_0}^{K}\frac{a_k}{S_k},$$

and the result follows.

• Part (*2*). Since $\sum\frac{a_k}{S_k(\log(S_k+1))^2} < \sum\frac{a_k}{S_k(\log(S_k))^2}$, it is enough to prove convergence of the second series. Now, by (2),

$$\frac{a_k}{S_k} < -\log\left(\frac{S_{k-1}}{S_k}\right) = \int_{S_{k-1}}^{S_k}\frac{1}{t}dt$$

and so, summing for $k > 1$,

$$\sum_{k\leq K} \frac{a_k}{S_k(\log(S_k))^2} < \sum_{k\leq K} \int_{S_{k-1}}^{S_k} \frac{1}{t\log(t)^2}dt = \int_{S_1}^{S_K} \frac{1}{t\log(t)^2}dt < +\infty$$

The result follows.

3 Examples

Corollary 3 $\sum_{k\leq K} \frac{1}{k}$ *diverges*

Proof: Trivial from Theorem 2 and the divergence of $\sum_{k\leq K} 1$.

Corollary 4 *The series* $\sum_n \frac{n^n}{n!e^n}$ *is divergent.*

Proof: For any $n \geq 1$, the inequality

$$e < (1+\frac{1}{n})^{n+1},$$

follows from (2) with $x = \frac{1}{n+1}$. Hence $\frac{(n+1)^{n+2}}{(n+1)!e^{n+1}} > \frac{n^{n+1}}{n!e^n} > \cdots \geq \frac{1}{e}$, and so the series $\sum_n \frac{n^{n+1}}{n!e^n}$ is divergent. Moreover, $S_n = \sum_{j=1}^n \frac{j^{j+1}}{j!e^j} > \frac{n}{e}$, and so

$$\sum_n \frac{n^n}{n!e^n} > \frac{1}{e}\sum_n \frac{n^{n+1}}{n!e^n S_n}.$$

The result now follows by Theorem 2.

Corollary 5 *Let* $f'(t) \geq 0$ *a decreasing function, and* $f(0) > 0$. *Then,* $\sum f'(n)$ *diverges if and only if* $\sum \frac{f'(n)}{f(n)}$ *diverges. Moreover,* $\sum \frac{f'(n)}{f(n)(\log(f(n)+1))^2}$ *always converges.*

Proof: Again, in the case when $\sum f'(n)$ is convergent, both results are trivial by noting that $f(n) \geq f(0)$, so we will assume $\sum_{n\leq N} f'(n) \to \infty$ with N. Let us prove the first part of Corollary 5. Now, since

$$S_n = \sum_{1\leq j\leq n} f'(j) < \int_0^n f'(t)dt = f(n) - f(0) < f(n), \tag{3}$$

we deduce that $f(n) \to \infty$ with n. Moreover,

$$S_n > \int_1^n f'(t)dt = f(n) - f(1) > \frac{1}{2}f(n),$$

for n sufficiently large. Hence,

$$\sum_{n\le N}\frac{f'(n)}{S_n}<2\sum_{n\le N}\frac{f'(n)}{f(n)},$$

and the result follows from Theorem 2.

The second part of Corollary 5 follows from the second part of Theorem 2 and (3).

Let $\log_1(x)=\log x$, and for any integer j, $\log_{j+1}(x)=\log(\log_j(x))$.

Corollary 6 *For any integer J, $\sum_k\frac{1}{k\prod_{j\le J}\log_j k}$ is divergent. On the other hand $\sum_k\frac{1}{k\prod_{j\le J}\log_j k(\log_{J+1}k)^2}$ is convergent.*

Proof: In Corollary 5, take $f_J(t)=\log_J(t)$. Note that for any $f_J(t)=\log f_{J-1}(t)$ we have $f'_J(t)=\frac{f'_{J-1}(t)}{f_{J-1}(t)}$. Now, Since $f'_J(t)=\frac{1}{t\prod_{j\le J-1}\log_j t}=\frac{f'_{J-1}(t)}{f_{J-1}(t)}$, we just have to use Corollary 3, Corollary 5, and apply induction. For the second part, use the second part of Corollary 5, (note that for any J and t sufficiently large depending on J, $\log_J t>\frac{1}{2}\log_J(t+1)$).

We include one final example just to show the wide range of applications of this criterion. We will use it to give a new proof of a well known fact in Analysis, consequence of $(L^2)^*=L^2$.

Corollary 7 *Let (X,μ) an space of measure with $\mu(X)<\infty$. Suppose $f:X\to\mathbb{R}$ is a measurable function such that*

$$\int_X|fg|d\mu<\infty,$$

for any $g\in L^2(X)$. Then, $f\in L^2(X)$.

Proof: Without lost of generality we can assume $f\ge 0$. By taking $g=1$ we see that $f\in L^1(X)$. Let us call $A_k=\{x\in X\,:\,k\le f(x)<k+1\}$. Then

$$\sum_{k\ge 0}k\mu(A_k)\le\int_X fd\mu<\infty. \tag{4}$$

Now suppose $f\notin L^2(X)$. Then

$$\sum_{k\ge 0}(k+1)^2\mu(A_k)>\int_X f^2d\mu=\infty,$$

and so, by (4)

$$\sum_{k\ge 0}k^2\mu(A_k)=\infty.$$

Now, let us call $S_k = \sum_{j=1}^{k} j^2 \mu(A_j)$, and consider $g(x) = \frac{k}{S_k}$ for any $x \in A_k$. Then $g \in L^2(X)$ since

$$\int_X g^2 d\mu = \sum_{k \geq 0} \frac{k^2}{S_k^2} \mu(A_k) < \infty$$

by the second part of Theorem 2, (note that $S_k > (\log(S_k + 1))^2$ for k sufficiently large), meanwhile

$$\int_X fg d\mu > \sum_{k \geq 0} \frac{k^2}{S_k} \mu(A_k) = \infty,$$

by the first part of Theorem 2. Hence, we get a contradiction and the result follows.

Clearly both, Theorem 2 and Corollary 5, seem to have a wide variety of applications, and we leave to the interested reader to find new ones.

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Referenser

[1] N. H. Abel, Crelle, Vol 3, p. 81, 1828.

[2] B. Demidovich, Problemas y Ejercicios de Análisis Matemático, Paraninfo, 1978.

[3] U. Dini, Sulle serie a termini positivi, Anali Univ. Toscana, Vol. 9, 1867.

[4] K. Knopp, Theory and application of infinite series, Dover, 1990.

[5] M. Spivak, Calculus, Cambridge University Press, 2006.

[6] W. Rudin, Análisis real y complejo, Mc Graw-Hill, 1988.

[7] Ch.-J. de la Vallée Poussin, *Cours d'Analyse infinitŐsimale*. Louvain: Librairie universitaire; Paris: Gauthier-Villars, 1954.