Normat **60**:2, 82–90 (2012)

A tale about tails

Tails as tools – tails as toys

Haakon Waadeland

haakonwa@math.ntnu.no

1 Apology.

As you will see rather quickly, this note is not a proper math paper, not even a survey paper. Almost nothing is new.

It is meant as a final Goodbye to my former colleagues and still friends – with my warm gratitude.

2 Concepts.

We deal with *continued fractions*

$$\mathop{\mathrm{K}}_{n=1}^{\infty}(a_n/b_n) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} =: \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots, \tag{1}$$

where a_n and b_n are complex numbers such that all *approximants*

$$S_N(0) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_N}{b_N} = \frac{A_n}{B_n} \tag{2}$$

are meaningful. The value ∞ is accepted. The continued fraction converges iff $\lim_{n\to\infty} S_n(0)$ exists. The limit is the value of the continued fraction. Again the value ∞ is accepted.

The numbers A_N and B_N, the way they are normalized, are known to satisfy certain linear recurrence relations, see e.g.[2] and [3] .

A generalization of (2) is

$$S_N(u) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_N}{b_N + u} = \frac{A_N + A_{N-1}u}{B_N + B_{N-1}u}, \tag{3}$$

where u is a complex number.

The concept of *tail* is essential in the theory of continued fractions. For a fixed N the continued fraction

$$\frac{a_{N+1}}{b_{N+1}} + \frac{a_{N+2}}{b_{N+2}} + \cdots \tag{4}$$

is called the *Nth tail* of the continued fraction (1), convergent or not. In all newer books on continued fractions tails and their applications as well as their role in the theory of continued fractions are presented. The deepest presentation is given in [4].

In case of convergence, if we in (3) choose $u = u_N =$ the value (4) of the tail of (1), then $S_N(u_N)$ is the value of the continued fraction (1). This is actually the key to the use of tails as a tool in continued fractions: Remove the unknown and correct Nth tail of the continued fraction (1) (like we do to get the *Nth approximant*) and replace it by a new tail, incorrect but in some sense "near" the correct tail. We then get what is called a *modified approximant*, which in favorable cases is better than the classical one. We now illustrate this in two simple cases. First, however, a little remark on continued fractions: A continued fraction $\mathrm{K}(a_n/b_n)$ may, if all a_n, b_n are different from 0 , be transformed to a fraction $\mathrm{K}(c_n/1)$ or $\mathrm{K}(1/d_n)$ in a simple way, see e.g. [2],[3] or [4].

3 Tail as tool.

Example 1.

Take the continued fraction

$$\frac{12+0.9}{1} + \frac{12+0.9^2}{1} + \frac{12+0.9^3}{1} + \cdots + \frac{12+0.9^n}{1} + \cdots. \tag{5}$$

It follows from several theorems, one of them the *parabola theorem* [2], [3], [4], that it converges. For $n = 5, 10, 20, 30$, we compute the values of the approximants. They are here presented with 10 true digits.

$$S_5(0) = 4.799987237 \qquad S_{10}(0) = 2.816847005$$
$$S_{20}(0) = 3.112991809 \qquad S_{30}(0) = 3.130817317$$

In order to get modified approximants we replace the tails in the given continued fraction by

$$\frac{12}{1} + \frac{12}{1} + \cdots, \tag{6}$$

which has the value 3. This gives us the modified approximants

$$S_5(3) = 3.151871068 \qquad S_{10}(3) = 3.129004674$$
$$S_{20}(3) = 3.131833817 \qquad S_{30}(3) = 3.131891251$$

Comparison of ordinary and modified approximants in this example illustrates how a skilled use of tails may accelerate the process of approximating a number by using continued fractions. (Information: The value of the given continued faction is 3.1318924157 with 11 true digits,) Let X denote the value of the continued fraction (5): We find in the computation

$$X - S_{30}(0) = 0.001075\cdots, \quad X - S_{30}(3) = 0.0000011\cdots$$

In the handbook [1] there is a large number of examples of highly non-trivial applications of tails to accelerate convergence, not only the one shown here, but extentions to more general methods, leading to striking results in convergence acceleration. Here we shall, however, restrict ourselves to one very simple example within the method we have already seen in the previous example. For more advanced examples we refer to [1].

Example 2

We shall use a story from old days to get a second example of use of tails. We go back to a paper from 1865 by Julius Worpitzky, published in Jahresbericht from Friedrichs Gymnasium und Realschule. Worpitzky's theorem is as follows, slightly adjusted for our present use: In the continued fraction

$$\frac{p}{1} + \frac{q}{1} + \frac{a_3}{1} + \frac{a_4}{1} + \frac{a_5}{1} + \cdots \tag{7}$$

let $p, q, a_3, a_4, a_5, \ldots$ all have absolute value $\leq 1/4$, with $=$ only in a finite number of cases. Then the continued fraction, as well as all the tails, will converge to some value in the disk $|w| \leq 1/2$. In the following we shall for simplicity assume that p and q both are real.

We now ask the following question, also raised in [3, p.36]: What can be said about the Worpitzky set of continued fraction values ω for fixed values of p and q? We have

$$\omega = \frac{p}{1} + \frac{q}{1+w}$$

where w can take any value in $|w| \leq 1/2$. A simple transformation leads to

$$w = \frac{(q+1)\omega - p}{p - \omega}.$$

The ω-set we are asking for is then

$$\left|\frac{(q+1)\omega - p}{p - \omega}\right| \leq \frac{1}{2}.$$

Standard procedure (including a bit of work) leads to the result, that the ω-set we are asking for is the disk

$$\left|\omega - \frac{p(4q+3)}{4(q+1)^2 - 1}\right| \leq \frac{2|pq|}{4(q+1)^2 - 1}. \tag{8}$$

As a numerical example take (as in [3]) $p = -1/4$, $q = 1/8$. Then the disk is

$$\left|w + \frac{14}{65}\right| \leq \frac{1}{65}.$$

This tells us that if we use $-\frac{14}{65}$ as an approximation, then the error is less than $\frac{1}{65}$ for all continued fractions in question, i. e. all where $|a_n| \leq 1/4$ for all $n \geq 3$.

(A different numerical example is what we get with $p = 1/5, q = -1/6$. This leads to the disk $|\omega - 21/80| < 3/80$.)

4 Tail as toy

There are many different ways to play with tails. We start with an example of simplest kind.

Example 3.

Given a continued fraction. The game is to cut it off and replace the removed tail by a new one. In this example we take the simplest one of all, namely the 1-periodic continued fraction. Moreover, we assume that the elements are positive and that the starting continued fraction is a 1-periodic fraction:

$$u := \frac{x}{1} + \frac{x}{1} + \frac{x}{1} + \cdots = \sqrt{x + 1/4} - 1/2$$

For the purpose of a more "visible" computation we take

$$x = \frac{9}{64},$$

in which case we get

$$u = 1/8.$$

A tail to be used, to replace the removed tail, comes in this example from a continued fraction

$$y = y(t) = \frac{t}{1} + \frac{t}{1} + \frac{t}{1} + \cdots = \sqrt{t + 1/4} - 1/2,$$

also 1-periodic with positive elements. The "result of the game" may then be illustrated e.g. as folllows

$$\frac{x}{1} + \frac{x}{1} + \frac{x}{1} + \frac{x}{1} + \frac{x}{1} + \frac{t}{1} + \frac{t}{1} + \frac{t}{1} + \cdots.$$

More generally

$$\frac{c_1}{1} + \frac{c_2}{1} + \cdots + \frac{c_N}{1} + \frac{d_1}{1} + \frac{d_2}{1} + \cdots. \tag{9}$$

As we can see, a tail is replaced by a tail of another continued fraction, like we do to accelerate convergence, see Example 1. But here the purpose as well as the follow-up is a different one.

5 A trip to the garden.

What we are doing here is in a strange way related to what a gardener does, when he on a rowanberry tree grafts a branch from an apple tree. With some luck the rowanberry tree with an apple tree branch will give (on that branch) apples. Some gardeners play with trees and branches in the same way as we play with continued fractions. (In (9) the set of numbers c_n corresponds to the rowanberry tree, whereas the d-set corresponds to the apple tree.) Back to math!

Here we shall place the new tail in turn: after the second, after the third, and after the fifth term in the given continued fraction. Keeping the numerics we shall call the (value of) the new continued fraction $T(N,t)$. For $t = x = 9/64$ the t-fraction $y(t)$ takes the value $1/8$, and $T(N, 9/64) = 1/8$ for all vales of N..

For $t = 0$ we have $T(N,0) = S_N(0)$, the nth approximant of the given continued fraction. It is a consequence of the theory of continued fraction with postive elements thst here $S_N(0) < 1/8$ for $N = 2$ and $S_N(0) > 1/8$ for $N = 3$ and for $N = 5$.

We list the three cases as below, with $t = 0$, with $t = 9/64$ and with $t = \infty$ (implying $y = \infty$).

$N = 2$

$$T(2,t) = \frac{x}{1} + \frac{x}{1+y},$$

leading to

$$T(2,t) = \frac{x(1+y)}{1+y+x} = \frac{9(1+y)}{73+64y},$$

and hence:

$$\begin{aligned} t = 0 :& \qquad T = 9/73 = 0.123287671 \\ t = 9/64 :& \qquad T = 1/8 = 0.125 \\ t = \infty :& \qquad T = 9/64 = 0.140625 \end{aligned}$$

$N = 3$

$$T(3,t) = \frac{x}{1} + \frac{x}{1} + \frac{x}{1+y}$$

leading to

$$T(3,t) = \frac{x(1+x)+xy}{1+2x+(1+x)y} = \frac{9}{64}\frac{(73+64y)}{(82+73y)}$$

and hence:

$$\begin{aligned} t = 0 :& \qquad T = 0.125190548 \\ t = 9/64 :& \qquad T = 0.125 \\ t = \infty :& \qquad T = 0.140625 \end{aligned}$$

$N = 5$

$$\frac{x}{1} + \frac{x}{1} + \frac{x}{1} + \frac{x}{1} + \frac{x}{1+y},$$

leading to (after some rearrangements) the following expression

$$\frac{x(1+3x+x^2)+y(x+2x^2)}{(1+4x+3x^2)+y(1+3x+x^2)}.$$

The fifth approximant of the non-teminating fraction, evaluated at the x-value given above, is then

$$\frac{x(1+3x+x^2)}{1+4x+3x^2} = \frac{9.5905}{6643.64} = 0.125002352$$

Hence $t = 0$: $T = 0.125002352$.

Moreover, as always: $t = 9/64$: $T = 1/8$.

Finally $y = (t) = \infty$ leads to the value

$$\frac{(x+2x^2)}{(1+3x+x^2).}$$

We substitute $x = 9/64$ and get

$$T(5,\infty) = \frac{9 \cdot 82}{59059} = 0.124978831.$$

Remark. We observe that the variation between the values of the continued fractions created by the game is rather small. This goes back to the way the continued fractions are chosen. A brief presentation of a quite different situation follows.

Given the continued fraction in Example 1, and let the "interrupting" continued fraction be as the $y(t)$ from Example 3. With the same notations as before we get

$$T(2,t) = \frac{12.9}{1+\frac{12.81}{1+y}},$$

leading to (since $y(0) = 0, y(12) = 3$)

$$T(2,0) = 0.9341..., T(2,12) = 3.0606..., T(2,\infty) = 12.9.$$

$$T(3,t) = \frac{12.9}{1} + \frac{12.81}{1} + \frac{12.729}{1+y}$$

$$T(3.0) = 6.67335‘ \qquad T(3.12) = 3.1744440 \qquad T(3,\infty) = 0.9341$$

6 A contribution from the real world

In the previous example the functions or continued fractions were picked merely to serve the purpose of being toys in a game with continued fractions. In the example to come here the function and its continued fraction expansion are picked right out of the middle of the mathematics everyday toolbox. We have here used a rather well known (but not *very* well known) continued fraction expansion of the confluent hypergeometric funcion being equal to the exponential function

$$\exp(z) = {}_1F_1(1;1;z) = \frac{z}{+2} \, \frac{}{+} \, \frac{z}{-3} \, \frac{}{+} \, \frac{z}{2} \, \frac{}{+} \, \frac{z}{-5} \, \frac{}{+} \, \frac{z}{2} \, \frac{}{+} \, \frac{z}{-7} \, \frac{}{+} \cdots$$

This function (of z) is, as well known, analytic in the whole plane. Our game with this continued fraction is, however, such that we keep $z = 1$ fixed. As long as nothing else is changed it has the value e. Our game is to replace all numbers 2 by a variable u, and study the function

$$g(u) := \frac{1}{1} \, \frac{}{+} \, \frac{1}{-1} \, \frac{}{+} \, \frac{1}{u} \, \frac{}{+} \, \frac{1}{-3} \, \frac{}{+} \, \frac{1}{u} \, \frac{}{+} \, \frac{1}{-5} \, \frac{}{+} \, \frac{1}{u} \, \frac{}{+} \, \frac{1}{-7} \, \frac{}{+} \cdots$$

Here we use an old theorem by Sleszinsky–Pringsheim: The continued fraction $\mathrm{K}(a_n/b_n)$ converges if for all n

$$|b_n| \geq |a_n| + 1.$$

In our case all $a_n = 1$. Hence, if $|u| \geq 1$ the continued fraction converges. Deeper results can be proved, but we will not go into that here.

We will be aiming at asymptotic properties, and replace $g(u)$ by $w * g(1/w)$. For that purpose we study the function $w * g(1/w)$ in a neighborhood of the origin $w = 0$.

A proper discussion of the power series shows that approximation number 2 gives

$$wg(1/w) = 1 + \frac{2}{3}w + \frac{1}{9}w^2 + O(w^3)$$

Going back to u again,we find the relation

$$g(u) = u + \frac{2}{3} + \frac{1}{9u} + O(u^{-2})$$

Geometric interpretation of the asymptotic investigation:

Theorem 1 *The graph of the function $g(u)$ approaches the straight line*

$$g = u + \frac{2}{3}$$

when $u \to \infty$.

Approximation number 5 leads to

$$g = u + \frac{2}{3} + \frac{1}{9u} - \frac{2}{135u^2} - \frac{1}{405u^3} + \frac{2}{1701u^4} + O\Big(\frac{1}{u^5}\Big).$$

7 Crossing the border. A tail used as a passport

Continued fractions of the form

$$\frac{F_1 z}{1+G_1 z} + \frac{F_2 z}{1+G_2 z} + \frac{F_3 z}{1+G_3 z} + \cdots \tag{10}$$

are called Thron fractions, or simply T-fractions. Here F_k and G_k are complex constants and z a complex variable. The Thron fractions have several interesting properties, in particular on applications, see [1].

In the present section we shall restrict ourselves to a very special example:

$$\frac{z}{1-z} + \frac{z}{1-z} + \frac{z}{1-z} + \cdots \tag{11}$$

Simple recursive computation shows that the nth approximant is

$$\frac{z\big(1-(-z)^n\big)}{1-(-z)^{n+1}} \tag{12}$$

This shows that in $|z| < 1$ the continued fraction by convergence defines the function

$$f(z) := z, \tag{13}$$

whereas in $|z| > 1$ it defines the function

$$g(z) := -1. \tag{14}$$

Take any approximation and modify it with the relevant tail value. Keep in mind that the two formulas for the approximation, (11) reduced to the first n terms and 12 as it stands, are equal formulas. This is true for g as well as for f.

Modification of f.

In the "(11)-version" of the approximation, add in the last denominator the proper value of the tail, which is z. Then the whole expression "telescopes" down to the term z, valid in the whole plane, except at $z = \infty$. We thus have analytic continuation of f to the whole plan minus ∞ .

$$z = \frac{z}{1-z} + \frac{z}{1-z} + \cdots + \frac{z}{1-z+z}$$

Modification of g. As above, except that the proper tail value now is -1. The telescoping will here lead to the value -1, which is now the analytic continuation of g to the whole Riemann sphere minus the origin $z = 0$.

$$-1 = \frac{z}{1-z} + \frac{z}{1-z} + \cdots + \frac{z}{1-z-1}$$

References

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