Reflections on a CF-expansion of $2 + 2^{1/3}$

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1 Introduction

Continued fractions are essential in the paper. An informal presentation of how they are written is as follows:

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} =: \frac{a_1}{b_1} \,+\, \frac{a_2}{b_2} \,+\, \frac{a_3}{b_3} \,+\, \cdots$$

If the a_n's , b_n's or both are again continued fractions, we get a *branched* continued fraction.

The background for the present paper is a branched continued fraction from our handbook [1], representing the number $2 + 2^{1/3}$. In the work with the handbook the formula (1) below was found on Internet. But later, in the concluding work on the handbook, the formula was no longer there. Nevertheless, we included it (although reluctantly). Some weeks after publication of the book, we got a letter from Domingo Gomez in Venezuela, asking us where we had found the formula. We told him about Internet, and expressed our apologies for the lack of references. He then sent a new letter, containing among other things, a reference to [2], which we should have seen, since we already had that book.

The following result, presented on page 184 in [1] is found in [2], [3]:

The number $2 + 2^{1/3}$ can be represented by a branched continued fraction: The equation

$$C = 3 + \frac{1}{3} \,+\, \frac{C}{3} \,+\, \frac{C}{3} \,+\, \frac{C}{3} \,+\, \cdots, \tag{1}$$

has $C = 2 + 2^{1/3}$ as a solution. In (1) we replace on the right hand side repeatedly C by the same continued fraction, then

$$2 + 2^{1/3} = C = 3 + \frac{1}{3} \,+\, \frac{3 + \frac{1}{3} + \frac{C}{3} + \frac{C}{3} + \cdots}{3} \,+\, \frac{3 + \frac{1}{3} + \frac{C}{3} + \frac{C}{3} + \cdots}{3} \,+\, \cdots.$$

The first approximants are

$$C_0 = 3, \quad C_1 = 3 + \frac{1}{3}, \quad C_2 := 3 + \frac{1}{3} {}_{+} \frac{C_0}{3},$$
$$C_3 := 3 + \frac{1}{3} {}_{+} \frac{C_1}{3} {}_{+} \frac{C_0}{3} = 3 + \frac{1}{3 + C_1(C_2 - 3)},$$

and generally

$$C_n := 3 + \frac{1}{3} {}_{+} \frac{C_{n-2}}{3} {}_{+} \cdots {}_{+} \frac{C_0}{3} = 3 + \frac{1}{3 + C_{n-2}(C_{n-1} - 3)}.$$

Numerical values of the first approximants:

$$C_0 = 3, \quad C_1 = 3 + \frac{1}{3} = 3.3333\ldots, \quad C_2 = 3 + \frac{1}{4} = 3.25, \quad C_3 = 3 + \frac{6}{23} = 3.2608\ldots$$
$$C_4 = 3.2598870\ldots, C_5 = 3.25991189\ldots, C_6 = 3.25992366\ldots$$

Since $C = 3.25992104989\ldots$ we see that already the approximants of order 4, 5, 6 are very good.

2 Why $a = 3$?

A natural (and vague) question here is: Let a be a positive number. In the formula

$$C = a + \frac{1}{a} {}_{+} \frac{C}{a} {}_{+} \frac{C}{a} {}_{+} \frac{C}{a} {}_{+} \cdots \tag{2}$$

is $a = 3$ the only value for which we get such a nice result? In case of YES, then WHY? An attempt to come up with a possible answer starts with the observation that the continued fraction (2) converges for all C outside the ray $(-\infty, -a^2/4)$. In the following we assume that C is located outside this ray. Actually, we are looking for *positive* solutions C. Let S be the value of the continued fraction

$$S := \frac{C}{a} {}_{+} \frac{C}{a} {}_{+} \frac{C}{a} {}_{+} \cdots. \tag{3}$$

Then we have

$$S := \frac{C}{a + S},$$

leading to

$$S := -\frac{a}{2} + \sqrt{C + \frac{a^2}{4}},$$

and hence, from (2)

$$C := a + \frac{1}{\frac{a}{2} + \sqrt{C + \frac{a^2}{4}}}. \tag{4}$$

Observe that any solution C has to be $\geq a$. A rearrangement followed by a squaring gives

$$\big(\frac{1}{C-a}-\frac{a}{2}\big)^2 = C+\frac{a^2}{4},$$

and finally

$$\frac{1}{(C-a)^2}-\frac{a}{C-a}-C=0.$$

For $C \neq a$ we get the cubic equation

$$H := C^3 - 2aC^2 + (a+a^2)C - 1 - a^2 = 0. \tag{5}$$

The solution is well known already from the time of the renaissance (Cardano and others). We let MAPLE do the work for us and describe the result as follows: With

$$P := \Big(36a^2 - 8a^3 + 108 + 12\sqrt{-3a^4+54a^2+81}\Big)^{1/3}, \quad Q := \Big(\frac{1}{3}a - \frac{1}{9}a^2\Big). \tag{6}$$

the roots are:

$$x_1 = \frac{P}{6} - \frac{6Q}{P} + \frac{2a}{3}, \quad x_2 = \frac{-P}{12} + \frac{3Q}{P} + \frac{2a}{3} + \frac{\sqrt{3}}{2}I\Big(\frac{P}{6}+\frac{6Q}{P}\Big);$$
$$x_3 = \frac{-P}{12} + \frac{3Q}{P} + \frac{2a}{3} - \frac{\sqrt{3}}{2}I\Big(\frac{P}{6}+\frac{6Q}{P}\Big);$$

Following Maple we use I instead of i. For $a = 3$ we have $Q = 0$, and the roots are very simple. We have

$$P := 6 \cdot 2^{(1/3)},$$

and hence the roots are

$$2^{(1/3)} + 2, \quad e^{(2I\pi/3)} \cdot 2^{(1/3)} + 2, \quad e^{(4I\pi/3)} \cdot 2^{(1/3)} + 2.$$

Only the first one is of interest to us. This gives an answer to the two questions raised. The two additional solutions are irrelevant for us. By going to other types of continued fractions similar questions may be studied.

3 Another example.

For a cubic equation with real coefficients like the equation (5) there are different cases to study: one real root and two complex conjugate roots, three real roots, where in special cases two or three may coincide.

The positive solutions $> a$ of (5) are possible C-values in (2), but these are by far not as nice as in the case by Domingo Gomez. We are not going into a discussion of the different cases. We choose as an example the case where the factor of I is 0, i.e. when

$$\frac{P}{6}+\frac{6Q}{P}=0,$$

or

$$P^2 + 36Q = 0, \tag{7}$$

or, in terms of a:

$$Z := \left(36a^2 - 8a^3 + 108 + 12\sqrt{-3a^4 + 54a^2 + 81}\right)^{(2/3)} + 12a - 4a^2.$$

The solutions R of the equation $Z = 0$ give the a-values for which $x_2 = x_3$. We find

$$a = R := \sqrt{9 + 6\sqrt{3}}, -\sqrt{9 + 6\sqrt{3}},$$

and stick to the positive value

$$R = \sqrt{9 + 6\sqrt{3}} = 4.403669476 \tag{8}$$

in the following. We find

$$x_1 = \frac{P}{6} - \frac{6Q}{P} + \frac{2a}{3} = 1 + \sqrt{3} + \left(1 - \frac{\sqrt{3}}{3}\right)\sqrt{9 + 6\sqrt{3}} \quad = \quad 4.593260526 \tag{9}$$

and

$$x_2 = x_3 = \frac{-P}{12} + \frac{3Q}{P} + \frac{2a}{3} = -\frac{1}{2}\left(1 + \sqrt{3}\right) + \frac{1}{2}\left(1 + \frac{\sqrt{3}}{3}\right)\sqrt{9 + 6\sqrt{3}} \quad = \quad 2.107039213 \tag{10}$$

The equation (5) is thus satisfied with $a = R$ from (8) and $C = x_1$ from (9) or $C = x_2 = x_3$ from (10). Only the first one is a solution of (2):

$$C = a + \frac{1}{a} \, {}_{+} \, \frac{C}{a} \, {}_{+} \, \frac{C}{a} \, {}_{+} \, \frac{C}{a} \, {}_{+} \, \cdots$$

with $a = \sqrt{9 + 6\sqrt{3}}$ and $C = 1 + \sqrt{3} + \left(1 - \frac{\sqrt{3}}{3}\right)\sqrt{9 + 6\sqrt{3}}$.

4 A short remark on a different approach

For a fixed positive a we write the equation (5) in the following way:

$$(C - a)^2 + a = \frac{1 + a^2}{C}$$

To make notation more familiar we replace C by x and the two expressions by y. We can then solve the equation graphically in the following way:

$$y = (x - a)^2 + a \quad \text{(Parabola)}$$
$$y = \frac{1 + a^2}{x} \quad \text{(Hyperbola)}$$

The x-values of the intersection of the two curves are the C-values we are looking for.

References

1. A.Cuyt, V.Brevik Petersen, B.Verdonk, H.Waadeland, W.B.Jones, *Handbook of Continued Fractions for Special Functions*, Springer, 2008.

2. Steven R. Finch, *Mathematical Constants, p. 3-4*, vol 94 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 2003.

3. Domingo Gomez Morin, *La Quinta Operacion Aritmetica,* ISBN: 980-12-1671-9