Lexell's theorem

Ulf Persson

Matematiska Institutionen
Chalmers Tekniska Högskola och
Göteborgs Universitet
ulfp@chalmers.se

Introduction

Given a line segment AB in the plane, and consider all triangles with AB as a base and with fixed area. What is the locus of the opposite vertex? This has an obvious answer, as all those triangles must have the same height h. The answer is hence the line parallel to that given by the base and hence with a fixed distance from it.

Lexell's theorem states that in the case of a sphere, we should replace the line with a (small) circle. I.e. the intersection with the sphere of a plane, not necessarily passing through the center of the sphere (the latter case corresponding to a great circle, i.e. a geodesic on the sphere).

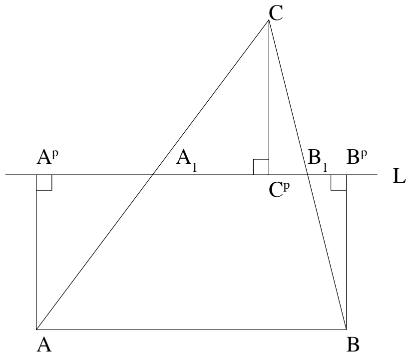
At first one may be a bit puzzled. Think of a circle in the plane and a segment AB outside it. For some points C on the circle the triangle ABC is bound to intersect the circle, meaning that part of its circumference is contained in the interior of the triangle. For any point C' on such an arc the triangle ABC' is clearly properly contained in ABC and hence has strictly smaller area. However, by choosing points A', B' such that AA', BB' are tangent to the circle, they determine an arc, points C on which, determine triangles ABC which only intersect the circle in C. The same argument holds for any convex area in the plane. However, the plane picture is misleading, on a sphere this does not necessarily happen.

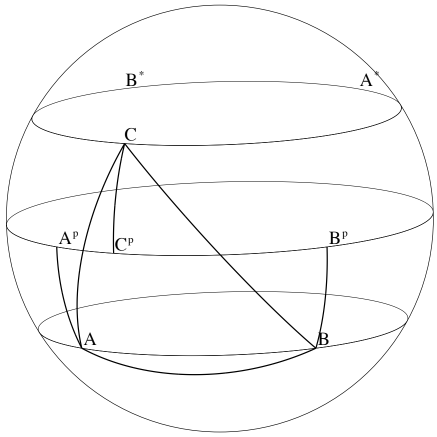
An exercise in Absolute Geometry

Consider the triangle ABC and let A_1, B_1 be the midpoints on the lines AC and AB respectively [1]. Let L be the line through them. Furthermore let A^p, B^p, C^p be the projections of A, B, C on L. Now the triangles AA_1A^p and A_1, C, C^p are congruent. (Two angles equal and the hypothenuse). The same argument works as well on BB_1B^p and B_1CC^p. From this we conclude that the area of the quadrilateral is equal to that of the triangle ABC, and that the distances of A and B to the line L are equal, and equal as well as to the distance of C to L. Note that we do not

[1] The following argument is referred to in 'On some classical constructions extended to hyperbolic geometry' by A.V.Akopyan, arxiv.org/pdf/1105.2153.pdf

use anywhere that the geometry is Euclidean, but it works as well in the spherical as hyperbolic case. We also note that the distances involved depend on the area of the triangle and are in fact uniquely determined by that. Thus if we fix the area of the triangle, the locus of C will form an equidistant curve to L. In the Euclidean case this simply means that the locus is a line parallel to L and hence the line AB, which as noted above could have been seen right away.





The Spherical Case

On the sphere the equidistant curves to a line L are given by the (small) circles cut out by planes parallel to the plane cutting out L. (Think of L as the equator, and the circles as the latitudes). This proves Lexell's theorem. By considering the anti-podal points A^*, B^* of A, B respectively, we can think of those curves as the pencil of circles through their base points A^*, B^*

Not so Fast!

Can this be true? What happens to the area of a triangle ABA^*? Could this take any value? Or if C varies close to A^* any small change of C leads to a great change in area. But if A and A^* are antipodal, there is an entire pencil of lines (great circles) joining the two points, and for each choice we get a triangle with its specific area. This also shows the extreme sensitivity to area, when the point C is chosen close to one of the anti-podal points of the end points of the base.

Bounds on areas of spherical triangles

Given any three points on a sphere, it determines in a sense two triangles, each being the complement of the other. Those have the same area only if the points lie on a great circle, and then it is given by 2π (half of the area of the sphere). The great circle is the largest circle through two points. The smallest is the one, having the segment as the diameter. A nice exercise for the reader would be to compute the area corresponding to the minimal circle.