On comparison and change

Leif Önneflod

Stora Hammar 301
370 42 Torhamn
leifskriv@telia.com

On Comparison and Change

Memory is short. The developers of modern mathematics once told us how to reason about mathematical operations. Now important parts of this are forgotten or misinterpreted. During the late 19th century efforts were made to bring the theoretical base for arithmetic to a closure. It ended up in a theory for logical manipulation of symbols. But the want for "pure abstraction" went alongside of a growing dissociation from reality. In the end we got proper theories for symbol expressions, but poor descriptions of how to lend structure to real-world mathematics. Accordingly a main issue in education is the difficulties with interpreting between math as experienced in everyday life and math as expressed by symbol expressions. They don't agree in a simple and straightforward way.

In this essay I will lay forward some informative models for central mathematical ideas and concepts. Those models take advantage from doing some deviations from the conventional way of writing and interpreting expressions. Nevertheless do those deviations agree better with modern mathematical theories, and concurrently with practical thinking, as well as the reasoning of the ancient Greeks. The way to perceive expressions will mutually support the perception of mathematical structures in real life.

What a comparison tells

We set off with the concept of comparison. Seemingly a trivial concept, and still an obstacle for pupils, when they meet word problems where comparisons are involved. Nowhere (in any textbook) I can find an in-depth analyse of the meaning and the consequences of comparison. The main road offered, to an understanding, is paved by myriads of examples.

The first thing to notice is that the answer to a (numerical) comparison comprises both a number and an *operation*. The latter is more or less "hidden" in the way we formulate the answer. Maybe we can accept that 'times' in the statement "One foot is twelve times bigger than one inch", is an operation. But what about the statement: "One foot compared to one inch is eleven inch more"? Or: "Three pounds are two pounds less than five pounds". Some acquaintance with Latin helps to find the operations in disguise. 'More' says 'plus' in Latin and 'less' says 'minus'! For convenience I introduce a symbol for 'compared to' – I suggest '※' – and write the last two comparisons as "1 foot ※ 1 inch = +11 inch" and "3 pound ※ 5 pound = = −2 pound". Here the plus and minus sign informs whether 'more' or 'less' applies. (Hence less need for words.)

In the same manner we can write "1 foot ※ 1 inch = ·12" for the first comparative statement above. Apparently comparisons can be made with different methods: an "additive" and a "multiplicative" one. To stress the difference, you may use the symbol '♀' for the latter comparison and write "1 foot ♀ 1 inch = ·12". The converse comparison is "1 inch ♀ 1 foot = /12". Here '/12' can be regarded as an 'unit fraction' (in Swedish: stambråk).

If this were all that there was to it, this would have been of minor use. However, such elements as '+11', '−2', '·12' and '/12', have a meaning on their own. They are more but mere abbreviations for words. For a while, we leave comparisons and pay our attention at elements of this kind.

Introduction of bound signs in integrated elements

It is a common notion, that an expression like $6 - 2 + 4$ can be rewritten as $(+6) + (-2) + (+4)$ (or as $6^{+} + 2^{-} + 4^{+}$), where the parentheses enclose positive and negative numbers. The problem is that those "numbers" have ambiguous interpretations to real-world situations.[1] The purpose of using parentheses, is to rule out subtraction and make the commutative and associative law apply.

There's another possibility of interpreting (+6) and (−2), but as positive and negative numbers. That is as an "increase by six", and a "reduction by two", respectively. In stead of 'increase' I will use the term 'addition', not in the confined mathematical sense, as it is known to us in Scandinavia, but in the more everyday sense, well-known to native English-speaking people. That is to say: here 'addition' ≡ 'increase'.

With $(+6)+(-2)+(+4)$ interpreted as a collection of additions and reductions, the perspective can change from having "numbers submitted to operations" to having "operational elements that compose". This change is crucial. As the former "numbers" are substituted for 'operating elements' like additions and reductions, the intermediate plus signs reduce in significance. They turn from active operations to passive signs for composition. As such, they can be omitted. This suggests we convert $(+6)+(-2)+(+4)$ into $+6\ -2\ +4$. Those new number elements, supplemented with what I denote 'bound' (appendant) leading signs, obey with ease the commutative and associative laws. E.g. $+6\ -2 = -2\ +6$, that is: no matter what order additions and reductions may come.

How then, to handle parentheses in the associative law? In expressions, fully equipped with parentheses as in $((+6)+(-2))+(+4) = (+6)+((-2)+(+4))$, the law works well, but what about "$(+6\ -2)\ +4 = \ldots$"? Neither $+6\,(-2\ +4)$, nor $+6\ -(2\ +4)$ will do the thing. As the purpose of using parentheses is grouping, and this now is directed at irreducible elements of change – not at numbers to be combined by operations yielding new numbers – we need another kind of parentheses submitted to that new kind of grouping. For reasons to be explained later on, the natural choice is 'square brackets'. A grouping like $[+6\ -2]$ yields, not a number,

[1] Some take this diversity for an advantage. Not so done by pupils. They often nurture an aspiration for single interpretations, to make life easier.

but a new element of change: $[+6\ -2] = +4$. By using such square brackets, the associative law works and will easily be understood: $[+6\ -2]\,+4 = +6\,[-2\ +4]$ i.e. $[+4]\,+4 = +6\,[+2]$, where the latter (redundant) brackets surrounding single 'changes' are used only to highlight their origin. In plain words: the order of grouping (additive) changes doesn't matter.

> The abstract concept of negative numbers, with one unique interpretation, does have some quite diverse interpretations into real world. Although this is not the *main* issue for this essay, subsequent sections will elucidate on the matter. The basic notion for "negative number" (or rather "minus term") will, in this article, be "reduction".
>
> Another issue: Operations are said to be binary. Numbers are seen as targets for operations. So can binary operations really be bound to numbers as if they were unary? Actually, it's a matter of choice. (After all, '+' and '−' are used as binary as well as unary operators.)
>
> In the established Category Theory, there are "morphisms", where sample expressions may be '+4', '·4', and '−1'. Morphisms can readily be combined by composition. Morphisms change (morph) 'objects',[1] for example points on a number line. Moreover, a translation of a point with '+4' (drawn as an arrow above the number line) is an example of 'group action' – in another discipline of modern mathematics. "Operating elements" are ubiquitous. There are, of course, mappings that out of necessity must be defined as binary. Most mappings *can* be done so, but that does not make it the one and only option.
>
> A binary operation of major importance is composition! That one will be used here.

Now back to "changes". It lies near at hand to name those additions and reductions 'plus terms' and 'minus terms' in conformity with the current designations. Those 'term elements' simplify more issues but the mere practise of the commutative and associative laws. For a start we can have a look at . . .

Order of operations

For expressions, we have "priority rules" stating the order of operations. Without those rules, we would have to tell the order by means of parentheses. Instead of $(3\cdot(5^2))+(4-(36/12))$ we can write $3\cdot 5^2+4-36/12$ applying the priority rules. Besides using those rules, a prerequisite to omit parentheses is that the associative law holds. This is commonly neglected. Take an expression like $4-3-2+1$. Only the first one of the five (separable) orders of calculation – illustrated by $((4-3)-2)+1$, $(4-(3-2))+1$, $(4-3)-(2+1)$,[2] $4-((3-2)+1)$ and $4-(3-(2+1))$ – leads to the intended answer. No wonder doing the sums "from left to right" is a widespread recommendation.[3] However, two kinds of problems soon surface:

- How to compute $4-x-2+1$ "from left to right"?

[1] I.e. not (literally) changing an object *into* another, but rather changing *from* an object *to* another. All in compliance with how 'mappings' are to be understood.

[2] This one is actually two "cases", depending on which expression within parentheses that is evaluated first.

[3] N.B. it follows the first sample of orders.

- How to perform computation shortcuts, as when cancelling the threes in $10 - 3 + 5 + 3$, or the x's in $10 - x + 5 + x$, if the only familiar order is "from left to right"?

Well, by adding still more "rules" (to be memorized), any problem can be settled. That's a safe horse, but not the one to be used here.

Now compare this to using "changes". No more do different orders of composition make any difference: $[[+4\ -3]\ -2]\ +1 = [+4\,[-3\ -2]]\ +1 = \ldots$
In addition, as the elements are trivially commutable, why not regroup like in $[+4\ +1][-3\ -2] = +5\ -5$? An added x does not raise the complexity level: $+4\ -x\ -2\ +1 = [+4\ -2\ +1]\ -x = +3\ -x$. An intervening x becomes no obstacle to compose remaining changes. A single "3 more" equals the affect of the group of "4 more, 2 less and 1 more" and may replace it.[1] The leading sign in each element tells **how the number affects the expression as a whole**. This is a crucial piece of understanding.

> That, what makes the order $((4-3)-2)+1$ of calculation work, and work the same as $[[+4\ -3]\ -2]\ +1$, is that no left parentheses break the bond between leading sign and trailing number. This too speaks in favour of "bound signs". The ordinary use of parentheses for regrouping purposes interferes in a complex way with the operational structure. (As in $(7-5)+4 = 7+(-5+4)$.) Regrouping of elements using square brackets maintain a stable element interpretation. The operation used *between* elements is 'composition' that is associative per se, N.B.

Real-world interpretation of terms

Let us have a closer look at the real-world correspondences to terms in expressions. The trivial interpretation is as additions and reductions. What has to be stressed upon is that in abstract expressions, this is the only conception of terms – they express additions and reductions in some common abstract unit.[2] There is yet another interpretation in real life. To make my point, I go back to antiquity.

The ancient Greeks observed that the only possible way to relate two quantities ($\alpha\rho\iota\theta\mu o\varsigma$ – an amount of units) was to add or subtract them, provided they had a unit in common. The Greeks only recognised numbers as showing the *magnitude* of some phenomenon: a number of feet, barrels, hours et cetera. The magnitude represented a quantity. Only magnitudes with the same unit could be related.[3] One way of relating was, so to say, to "compose" quantities by adding or subtracting them.

> Another way was to compare different *pairs* of (multiplicative) relations. They observed that 12 inch relates to 3 inch the same way 20 inch relates to 5 inch. If the two relations were "the same", they were said to be *analogous*; we had an analogy.[4] The fact that 20 is 4 times greater than 5, was expressed as (with u for some 'unit') "20 u is to 5 u *as* 4 u is to u". On purpose, I did not write '1 u', as the Greeks didn't consider '1' a number – '1' was the unit itself. One unit *is*

[1] This simple kind of reasoning is not accessible for quite a many pupils. When they become familiar to it, they show reactions like relief and inspiration. The reasoning can easily be practically experienced by learners via hands-on materials.

[2] Long since, already Aristotle pointed out that a unit can be abstract.

[3] Their thinking alone, would demand for a lengthy essay to be fairly described. This is the short-short version.

[4] Greek logos = relation, ana- = the same. The 'logos' was the 'ratio' relation.

> the unit, they reasoned. Nowadays we write “20 : 5 = 4 : 1” or “20 : 5 :: 4 : 1”. All calculations corresponding to multiplication and division were performed via “analogies”, but enough for that.

The statement “When relating (composing) quantities, the only operations to perform on them are addition and subtraction”, have a converse: “The only things that can be added or subtracted are quantities”. (Alas, this cannot be inferred in a few lines. The reader may accept this as an unproven postulate.) Well, how then to relate this to additions and subtractions? The abstract terms are meant to reflect changes. There is a frequent misconception that expressions mirror some course of events in real life. That is not so. The terms in an expression no more than mirror how the quantities in reality *contribute* to compute the answer to a *specific* posed question.

To make my point clear, let us imagine the scene of an empty bench. At 13:45 three people sit down. At 13:55 another two join. That makes them five. At 14:10 four of them leave. Now there is one left. At 14:15 the last one leaves.

Here are two samples of all possible questions to pose.

- How many more leaves at 14:10 than arrives at 13:55?
  This is computed by $4 - 2 = 2$. Here we subtract the people that *arrived* (= ‘more’) and those who *left*, corresponds to a “positive number”.
- Given that there are five people on the bench at 14:00 and two arrived at 13:55, how many were they before 13:55?
  This is computed by $5 - 2 = 3$. Here the 5 rightly is a positive number, but once again we subtract with the arrivals.

Those examples might have shown my point. The idea of “contribution” is crucial when to perform an interpretation from reality to an abstract expressions. It also leads to the second interpretation of terms. The following reasoning may show that:

How do three people and two people *contribute* to the number of people on the bench at 14:00? The answer is evident: they both *add* to the number of people! Seen through the “glasses of the contribution idea”, all quantities (subject to plain composition) correspond to plus terms. They *add* to the result. Thus, a quantity cannot be negative.[1] However, the contribution to a *computation* can be negative as well! So if Kim loses weight from 120 pounds by 7 pounds, the computation of the new weight can be written as $+120 \; -7 = +113$, where $+120$ is the quantity “before”, -7 is the quantity change and $+113$ is the quantity “after”.

To conclude: positive terms can represent quantities *or* changes in quantity, whereas negative terms only represent changes.[2] In its abstract general sense “change” merely depicts relations. In real life terms are associated with quantities. Terms are coherent (integrated) elements, that obey the associative and commutative laws with ease. Grouping and regrouping turn trivial.

Reality structured with the $\mathcal{RAY}$ model

As the next step to close in on comparisons, we have a look at the $\mathcal{RAY}$ model. The letters stand for $\mathcal{R}$oot or $\mathcal{R}$eference (= origin value), $\mathcal{A}$lteration (= change

[1] This was why the resistance to negative numbers was so fierce when they were introduced – nothing in real life could have a negative size.

[2] I leave out the topic of “directed quantities”.

value) and $\mathcal{Y}$ield (= result value). The model simply states that a Root subject to an Alteration results in a Yield.

For example: $+5\,\mathrm{kg} \xrightarrow{+3\,\mathrm{kg}} +8\,\mathrm{kg}$ follows the pattern of $\mathcal{R} \xrightarrow{\mathcal{A}} \mathcal{Y}$.

Here $\mathcal{R} = +5\,\mathrm{kg}$ and $\mathcal{Y} = +8\,\mathrm{kg}$ are quantities and $\mathcal{A} = +3\,\mathrm{kg}$ a quantity change. Writing the alteration over an arrow, emphasizes its role as an intermediate[3] between the "before" ($\mathcal{R}$) and "after" ($\mathcal{Y}$) component. Evidently, this models mathematical events in real life. In most arithmetical (single-step) computations, the values involved divide into the three categories of $\mathcal{RAY}$. The need to perform a computation arise when two values are known and the third is sought for. The $\mathcal{RAY}$ model offers a structure to the three constellations that the two known values and the sought value can form. The first two are, built on the example above, the following:

F. Forward computation $\mathcal{R} \xrightarrow{\mathcal{A}} ?$ e.g. $+5 \xrightarrow{+3} ?$

Problem illustration: "A dog weighs 5 kg. After half a year, it weighs 3 kg more. What is its weight then?"

B. Backward computation $? \xrightarrow{\mathcal{A}} \mathcal{Y}$ e.g. $? \xrightarrow{+3} +8$

Problem illustration: "A dog weighs 8 kg. It weighs 3 kg more than half a year earlier. What was its weight then?"

Evidently, the last problem is solved by $+8 -3 = +5$, but why does the alteration have to be reversed? The answer lies in the "forward alternative" figure. It demonstrates (by means of an arrow), that the alteration is applied on a value – a starting point – that is a root value $\mathcal{R}$. In the B alternative, the only value known to apply the alteration on, is the $\mathcal{Y}$ value. However, in order to apply $\mathcal{A}$ on $\mathcal{Y}$, the direction of the arrow has to be reversed. This is apparent from the figures above. Reversing the direction of the operation is coupled with reversing the operation itself. The $\mathcal{RAY}$ concept forwards the idea by help of the figure

$$? \underset{-3}{\overset{+3}{\rightleftarrows}} +8$$

where a reversed operation ('-3') accompanies the reversed lower arrow.

This mutual dependence between an operation and its direction, can easily be demonstrated to pupils and trained. I believe the reader can imagine how this can be achieved in practise.

How then, to transfer those alteration diagrams into ordinary expressions? Well, by applying the principle that an alteration follows after the value to be altered. For the "forward example", we write $+5 +3 = +8$. For the backward example, we write $+8 -3 = +5$, as the '-3' alters '$+8$', which is evident from the direction of the arrow.

Many pupils experience that problems, with this kind of reverse reckoning, are difficult to grasp. In the absence of a model like $\mathcal{RAY}$ (and thereby explicit instruction on how to identify the '$\mathcal{A}$' component that needs to be reversed) the pupil is left to her intuition as the only means to manage the problem.

[3] I.e. a mapping.

The choice to use the reverse operation becomes (unless equation techniques are used) more of an immediate choice, based on intuition. The reasoning, if any, is a reconstruction afterwards.

Separate inversion operator

The use of (operationally) integrated elements, with explicit operations, clears the way for directing inversions towards operations instead of numbers.① As an inversion of operation becomes an operation on its own, there is motive for introducing a separate operation, reserved for this single purpose. I suggest '∾' to be a general symbol for inverse operation. Using the $\mathcal{RAY}$ model we now can write …

forward computing as $\mathcal{RA} = \mathcal{Y}$

backward computing as $\mathcal{R} = \mathcal{Y} \backsim \mathcal{A}$ (or $\mathcal{Y} \backsim \mathcal{A} = \mathcal{R}$)

When solving the problem above we can write:

$$?\,+3 = +8 \quad \Longrightarrow \quad ? = +8 \backsim +3 = +8\ -3 = +5.$$

If we want to reverse an operation performed, in theory (abstract algebra), we cannot do that directly. The reason is that in group theory, a group is about *one* operation acting on a set of elements. Accordingly, subtraction cannot be the one operation, since it does not comply with the associative law. So how to reverse the adding of 3 to 5? Well, in theory, by adding the *inverse of 3* with respect to addition. The converse to $5 + 3 = 8$ will be $8 + (-3) = 5$. Here the *number* is inverted. However, as no one can comprehend how a group of eight people can be reduced by adding "three *negative* people" (however they might look), this is explained to be *the same* as subtracting, i.e. reducing, by three people. (It can be questioned whether something impossible can be "the same" as something quite natural.)

All the line, reversing (inverting) operations is simpler to comprehend. Compare using $\backsim/3 = \cdot 3$ instead of $(3^{-1})^{-1} = 3$ or $1/⅓ = 3$. Parallel the use of $\backsim -5 = +5$ to that of $-(-5) = 5$. Complex derivations, of how to create and calculate inverses, become superfluous. Explicit knowledge of opposite operation pairs will do.

Another point is that using subtraction to create "additive inverses", and division to create "multiplicative inverses", obfuscates the real-world structure of comparisons and backward computing. (The purpose not being to 'subtract? or 'divide?, but to *reverse* operation.) Above that, it adds an extra difficulty, as a subtraction by a number must be converted into a negative number, before an outer "subtraction" can be applied on the expression. Likewise, to perform a division of an expression holding a division by a number, the number must be converted into its inverse (with respect to multiplication). Those manipulations have no correspondences whatsoever in real-world, when we solve problems where inversions occur in the solution. Unlike this, direct inverting of the operations themselves, reflects the structure to be understood.②

① "Operation" has a dual reference to real actions and to symbols in expressions. You may tell which one by saying '*reverse* an operation' or '*invert* an operation', respectively. By substituting 'invert a number' for 'invert an operation' the established language is maintained.

② In abstract algebra, the use of inverting operators ($-$ and $^{-1}$) complies better with the structure of comparisons.

Another view on computing 'change'

Now we are set to attend to the third and last constellation that the known and sought values in $\mathcal{RAY}$ can form. It looks like $\mathcal{R} \xrightarrow{?} \mathcal{Y}$. For instance: $+5 \xrightarrow{?} +8$.

In the "dog example", the problem can read:

> "A dog weighs 8 kg. Half a year earlier, it weighed 5 kg.
> How much has it gained in weight?" (a)

The question has some alternatives as ...

> "How much less did it weigh then?" (b)
> "How much more does it weigh now?" (c)
> "How much does it weigh now compared to half a year earlier?" (d)
> "How much did it weigh half a year ago compared to now?" (e)

Perhaps your attention were drawn to the words 'less', 'more' and 'compared to'? The two first words hint at we are computing an alteration (change). This implies that a comparison is performed. Only the last two question alternatives, make the comparison stand out clear. Moreover, they shift the responsibility to judge the "sign" (more or less hidden in 'more' and 'less'), on to the pupil.[1] The $\mathcal{RAY}$ model helps to sort things out. As the answer to a comparison tells how to change a reference value (root, $\mathcal{R}$) to the value, which is to be described, it follows from the formula $\mathcal{R} \xrightarrow{\mathcal{A}} \mathcal{Y}$ that $\mathcal{Y} \backsim \mathcal{R} = \mathcal{A}$. (In the next section we see how the '※' comparison yields an '∾' inversion.) Applying that on the last two question alternatives, we get ...

for (d) question: $+8 ※ +5 = +8 \backsim +5 = +8 - 5 = +3$, i.e. "more".
for (e) question: $+5 ※ +8 = +5 \backsim +8 = +5 - 8 = -3$, i.e. "less".

Observe, that the order of elements in the comparing expression, strictly follows the wording (order) of the comparison.

When words like 'gained' in (a), 'less' (b) or 'more' (c) are given in advance, the pupil does not need to reflect on the structure of the comparison. The order of calculation will always be "the greater number subtracted by the lesser number".

> In the end, this lack of understanding strikes back. To get the sign right, as when calculating the slope of a line, an ability to apprehend a direction in comparisons is helpful. It is my experience, that this notion of "direction" seldom is accessible to the pupils as a working knowledge.

The fact that the comparison can be calculated by using the inverse of the root ($\mathcal{R}$), can, but of course, be stated as a rule and lent credit to by cases. Although, the apprehension that a comparison leads to the calculation of the $\mathcal{A}$ component in $\mathcal{RAY}$, gives us another option.

The two-step method for computing 'change'

To search for the $\mathcal{A}$ component means to search for what turns $\mathcal{R}$ into $\mathcal{Y}$. It takes two trivial steps to arrive at $\mathcal{Y}$. Firstly, we cancel $\mathcal{R}$ by applying $\backsim\mathcal{R}$. Secondly, we append $\mathcal{Y}$ itself. Altogether we have that $\mathcal{R}\,[\,\backsim\mathcal{R}\mathcal{Y}] = \mathcal{Y}$.

[1] Mind there is a direction in the comparison!

Example: $\mathcal{Y} ※ \mathcal{R} = +8 ※ +5 = [+8 ∾ +5]$,
since $+5\ [+8 ∾ +5] = [+5 ∾ +5]\ +8 = +8$.

Comparisons where "negative numbers" (i.e. reductions) are involved, don't differ in complexity level.

Example: $-7 ※ -2 = -7 ∾ -2 = -7 +2 = -5$, that is: "five less".
Interpretation: A reduction of 7 makes (something) reduce 5 more, than a reduction of 2. Quite obvious.

Here is a suggestion on how composition of terms may be learned: Perform the "changes" involved, on a positive quantity of sufficient size (which may be zero). A comparison between the size "before" and "after" the changes have been applied, reveals the total change. This is a winning concept in didactical situations. It is easily demonstrated by help of hands-on materials.

Example: Apply $[-7 +2]$ on $+10$. The result is $+5$. Finally, as $+5 ※ +10 = -5$, this implies that $[-7 +2] = -5$.[1]

I name this the "two-step method". In the world of arithmetical symbol expressions, this is no new phenomenon. The equation $ax = b$ has the solution ba^{-1}, [2]where you can identify ba^{-1} with $\mathcal{Y} ∾ \mathcal{R}$. The (possible) novelty is twofold:

1. An understanding of how to reason about comparisons in real life (where $\mathcal{RAY}$ is helpful).
2. A simplified universal method on how to construe the "inverse" by means of '∾'. (There is more to this.)

A more pictorial view of the two-step method can be obtained by starting from:

$$+a \xleftarrow{\;+a\;} +0 \xrightarrow{\;+b\;} +b$$

This shows how the quantities $+a$ and $+b$ are produced from a common origin in the center – the neutral quantity of $+0$. To compare $+a$ with $+b$, is the same as to find what transfers $+b$ into $+a$. By reversing the right arrow (and inverting the operation), you arrive at the two steps to take:

$$+a \xleftarrow{\;+a\;} +0 \xleftarrow{\;∾+b\;} +b$$

The reason to invert the operation for the comparison reference value $+b$, becomes obvious. Other "intermediate values" than $+0$ are possible. As to compare $+402$ with $+397$, the intermediate $+400$ can be an option. Incidentally, in Davydovs concept writing diagrams have proved to help learners get a "picture" of the problem.

More on scalar interpretations

To gain full benefit from the $\mathcal{RAY}$ model and the two-step method, one has to expand on the issue of real-world correspondences to integrated elements. So far we have come to know that ...

<u>The terms</u> correspond to quantities with units,

and to some extent that ...

[1] With hands-on materials, the obvious "circle" in this reasoning is broken, as the material provides the answer to the comparison, not yet another expression of a similar kind.

[2] Provided it is commutative, as holds for basic operations.

<u>The factors</u> correspond to *relations* between quantities.

Thus '$\cdot 3$' demonstrates the relation between a quantity of, say, $+3$ m and the unit $+1$ m. It also illustrates the relation between an infinity of other pairs of quantities like $+15$ s and $+5$ s or $+0.189$ kg and $+0.063$ kg. As the concept 'factor' is defined to refer to multiplication, the notion of 'scalars' seems to agree better to the use of both of the two opposite operators, as in $\cdot 5$ and $/5$.

In abstract algebra, rational numbers are defined using the 'equivalence class concept'. Pairs of integers e.g. (a, b) and (c, d) are said to be equivalent if $ac = bd$. (This is the old safe "golden rule" for proportionality in a modern outfit. Other equivalence conditions can be stated, but this is the one of interest for now.) Pairs that are equivalent all represent "the same" relation. Pairs equivalent to (a, b) form an 'equivalence class', denoted by $[(a, b)]$ or $[a, b]$. The equivalence class $[(3, 1)]$ will correspond to the scalar $[\cdot 3]$ as ...

- ▷ The identity scalar '$\cdot 1$' can be regarded as to implicitly provide the number '1' in the pair.
- ▷ The reference to multiplication is explicit and can be defined as to address the equivalence condition above.

Hence $[\cdot 3]$ can be considered an equivalence class. It comprises the infinite many relations between pairs of quantities. N.B. that the use of square brackets in equivalence expressions complies with this interpretation of $[\cdot 3]$. In the same manner $[/5]$ represents the equivalence class of $[(1, 5)]$ (number order reversed). Finally $[\cdot {}^a\!/\!_b]$ represents $[(a, b)] = [({}^a\!/\!_b, {}^b\!/\!_b)] = [({}^a\!/\!_b, 1)]$.

In the ancient Greek mathematics (due to Euclid), the relations between quantities (or 'magnitudes') always were expressed as pairs of quantities. (In a way, this is a fundamental kind of view to comprehend.) Their "analogies" have its modern parallel in equivalence classes.

On combining quantities and scalars

In order to combine quantities and scalars, we have to settle on two matters.

- The scalar primarily shows the *relation* between quantities. In practise, the scalar also can be interpreted as showing a "change" of one quantity into another. There is no contradiction between those two interpretations, but the first one has to be stressed upon. (I leave out the arguing in this matter.)
- According to the $\mathcal{RAY}$ model, a scalar is both the answer to a comparison between quantities ***and*** an element of quantity transformation (showing the relation). This duality of scalars is essential to recognize.

One more word about relations. – In symbol expressions, all signed term elements can be regarded as to show relations between quantities. A term representing a quantity, shows the relation to a zero quantity, $+0$. As already mentioned, all signed scalars show relations between quantities. A scalar being a measure,[1] shows the relation of a quantity to the unit quantity, $+1$.

The comparison of quantities, yielding scalars, was ascribed a symbol of its own: '♀', e.g. $\mathcal{Y} \mathbin{♀} \mathcal{R} = +18 \mathbin{♀} +6 = \cdot 3$. Here, the two-step method is employed to

[1] Observe, that 'measure' says 'mätetal' in Swedish.

construe the answer as follows: $+6\ [/6 \cdot 18] = [+6\ /6] \cdot 18 = +1 \cdot 18 = +18$. From this it is evident that $[/6 \cdot 18] = \cdot 3$ holds the answer to the comparison, as it supplies the transformation needed to change $+6$ into $+18$.[2]

The "pictorial view" of the two-step method may set off from:

$$+18 \xleftarrow{\cdot 18} +1 \xrightarrow{\cdot 6} +6$$

Here the unit $+1$ is used in the center, as it makes the factors above the arrows trivially equal to the measures. The next step is to "go from $+6$ to $+18$", which is accomplished by means of:

$$+18 \xleftarrow{\cdot 18} +1 \xleftarrow{\backsim \cdot 6} +6$$

Other intermediate quantities may be used as in:

$$+18 \xleftarrow{\cdot 6} +3 \xleftarrow{\backsim \cdot 2} +6$$

Moreover, this gives an alternative way to infer fractions (see later section) and to motivate why different fractions can work the same way. Still another example:

$$+5 \xleftarrow{\cdot 5} +1 \xleftarrow{\backsim /3} +{}^1\!/_3$$

... demonstrates the comparison, that conventionally is written as $5/\frac{1}{3}$. (The reader may picture the steps before and after.) The division by a fraction adds an extra difficulty to be unraveled. In contrast, inverting a division is no more difficult than inverting a multiplication.

> Diagrams as those above combine well with the use of e.g. cuisenaire rods.[1] Firstly, the pupils learn how to reason about comparisons and other matters by help of the rods. Next they learn to write diagrams to show the structure. Lastly they learn how this is written in mathematical expressions. Each step to be settled on, before advancing to the next.

Observe that the scalar transformation is written *after* the quantity to be transformed. E.g. $+24\ /8 = +3$; $+2 \cdot 5 = +10$ (and accordingly $+10 \ ♀ +2 = \cdot 5$). Hence all scalars need to have prefix sign notation.

> For a 'unit fraction scalar' like '/5', only a prefix division operator is possible. For historical and linguistic reasons an irregular use of the multiplication sign as "postfix" frequently occurs. (That is: the 'multiplicator' is put before the 'multiplicand', also referred to as 'premultiplying'.) This can become quite convenient at times, but here only the prefix notation will be used. (Also referred to as 'postmultiplying'.)[2]

Naturally, a leading quantity can be followed by many scalars, but (without using parentheses) only one leading quantity submits to its scalar successors. E.g. $+3 +5 \cdot 2 = +3[+5 \cdot 2] \neq [+3 +5] \cdot 2 = +8 \cdot 2$.

> This kind of relation is met with in different shapes in abstract algebra. E.g. vectors can be transformed by scalars. (Vectors also can be composed, i.e. added.)

[2] Pupils experience this model as quite conceivable and straightforward.

[1] Those are coloured rods of different lengths. They must only depict *quantities*, never scalars. It is important to discriminate between the two.

[2] I leave out the lengthy discussion, needed to fully justify the prefix preference, and to explain why and when postfix notation is an option. Some advantages with the 'prefix' alternative may still become apparent. After all, it's extensively used.

A general concept is 'group action', where a group of elements (here: scalars) by means of its operation, transform elements in another set (with another operation, if any at all). Quantities transformed by scalar elements, will be an example on 'group action'.

When I ask pupils to combine two scalars into a single one as in $x \cdot 12/3 = x \cdot a$ (a is asked for), they get puzzled.[1] They even hesitate about that $x \cdot 12/3$ will equal $x/3 \cdot 12$ (only reversed order of scalars). Apparently they have never been asked to compose "scalars". Such shortcomings lay the ground for trouble with algebra. Still, composition is easily understood, if the transform starting from '+1' come into use:

In order to compose the scalars $\cdot 3$ and $\cdot 2$, start from $+1[\cdot 3\ \cdot 2] = [+1\ \cdot 3]\ \cdot 2 =$ $= +3\ \cdot 2 = +6$. Then pose the question "What single scalar turns +1 into +6 ?". Apparently '$\cdot 6$' does the work, as $+1\ \cdot 6 = +6$. (Even this simple approach will turn out a "new way of thinking" to quite a lot.)

As even other combinations of elements (other than quantities and scalars) obey patterns very similar to those studied above, I introduce two concepts in order to ease generalization. The elements that perform the transformations (show the relations), I call **agents**. They *act* upon an opening element. The opening "target" for the agents I denote **anchor**, as it constitutes a fix starting point for the agents to act on.[2] They are, so to say, "tied up" to an anchor. This anchor is supposed to differ from the agents in some feature(s), e.g. kind of operation.

The anchor—agent relationship comes about in many shapes. In abstract algebra we have 'group action'. In education we meet concept pairs like 'multiplicand—multiplier' and 'dividend—divisor'. The multiplicand and dividend is always demonstrated as something with a size (and measure) that is subject to a (scalar) change by a multiplier and divisor respectively. It stands out clear that this has its exact correspondence in an anchor (target quantity) and an agent (scalar at work). The advantage using the anchor—agent concept is that it emphasizes a general kind of relation, equivalent to group action. The concept is applicable to powers (base—exponent relation) and yet another relation, both to be covered later on. This follows the principle of "Ockham's razor". It is rational to avoid an excess of concepts. (As in the 'multiplicand' family of concepts.)

The two kinds of division

The anchor and agent concept help to supply a theoretical setting to the didactical concepts of partitive and measurement division. In theory, we have but one kind of division, here exemplified by $12/3 = 4$. In real world we can differentiate between two kinds of division. The partitive division follows the idea of the dividend—divisor relation. With bound operations we can interpret this as $+12\ /3 = +4$, where $+12$ and $+4$ are magnitudes having a unit. Measurement division, on the other hand, is about computing a relation – "How many pieces of size 3 is contained by (goes into) the size 12 object?". The "division" is between measures of objects. The $\mathcal{RAY}$ model reveals that this follows the pattern of a comparison. The calculation will be $+12 \text{ ♀ } +3 = \cdot 12 \text{∾} \cdot 3 = \cdot 12\ /3 = \cdot 4$. Here we make two observations.

[1] Despite I don't use variables! It can look like $5\ \boxed{\cdot 12\ /3} = 5\ \boxed{}$, where the contents of the left hand box is to be substituted for *one* number and *one* operation in the right hand box.

[2] To be precise: 'opening' = 'leading' = positioned first.

1. The quantities of $+12$ and $+3$ have the measures of $\cdot 12$ and $\cdot 3$, which is an important thing to notice. It is ***not*** the quantities (with units) that we "divide", but their (scalar) measures. This is why we receive a scalar as an answer.

   This idea was stressed upon by the Oxford professor John Wallis, in the 17^{th} century. Due to Wallis, if the "genus" of the (quantity) number was left out, this allowed for scalar operations to take place between (remaining) "pure numbers". That is to say that "pure" scalar measures were separated from the quantity unit ("genus"). At that time the Euclid notion of numbers as quantities was prevalent, and the only ways permitted to compose quantities was to add or subtract them, or setting up analogies, i.e. equalling "ratios" between them. The multiplications and divisions that accompanied analogy computations, was regarded operations and did not infer that there were any independent numbers of the kind we nowadays identify with scalars. Wallis made way for this new idea.

2. The answer is a scalar. The size $+12$ holds the size $+3$ "four *times*", that is "$\cdot 4$". This is quite different from '$+4$' that tells the size of a third of size $+12$.

By means of bound operations we can make the difference, between the two ways of perceiving division, visible in all aspects.

> Frequently, the measurement division is written as a division between quantities. The quantity feature is apparent from the units supplied.
> For example: $\frac{10\,\text{apple}}{2\,\text{apple}} = 5$ or, as I would put it: $\frac{+10\,\text{apple}}{+2\,\text{apple}} = \cdot 5$

The idea of 'comparison' is so much more general than the idea of 'containment'. As a matter of fact, most divisions deal with comparisons in some way. I propose we substitute 'measurement division' for 'comparison division'. One advantage will be that the expression follows the wording order: "How much is 12 cm compared to 3 cm?" is written "$+12 \ ♀ +3$ " and so on. After some practise, one can go directly for the '$\cdot 12\,/3$ '. The idea that 'measurement division' computes a kind of measure, can be substituted for that a measure is computed by comparison between quantities (possibly to the unit). Another advantage shows when dealing with fractions, as will be seen.

'Level' – another real-world correspondence

Not only quantities can have a unit. We use numbers to denote different kinds of states or levels. I will use the term 'level' throughout. Examples may be water level; temperature; position (in a coordinate system); direction (e.g. in degrees); altitude (above sea level); date; point of time et cetera. Their measures all share the property of having *double references.* An ordinary quantity needs to reference nothing else but a unit. By comparing to the unit, the measure will be set. That will not do for 'levels'. A second reference to a "zero state" will be nescessary.

> Example: To determine a number (measure) for the altitude of a mountain, you have to settle on a zero altitude, usually chosen to be the sea level. In southern Sweden the "highest" mountain is 'Tomtabacken' ("Brownie Hill"), 378 m above sea level. I you go there, you won't be impressed – it is but a small hill rising about 20 m above the surroundings. On the other hand is Mauna Kea on Hawaii said to be the worlds highest mountain – measured from the nearby sea floor it rises from. It is apparent that the measure of "altitude" is a matter of how to choose, not only the unit, but also the reference level.

With a reference level established, the combination of the measure and the unit tells the "distance" from this level.

> A watch tells the "time elapsed" from midnight on. A coordinate tells the distance (and direction) from the origin (of coordinates).

The "double reference" is not the only feature of levels. Another is that they cannot compose. (e.g. get added). "What would '4$^{\text{th}}$ of July' + '14$^{\text{th}}$ of July' add up to? What is the meaning of the sum of two positions? (Ulf Persson wrote an article on this matter in 'Nämnaren' no 2, 2010.)

As those level elements *cannot operate* on each other (= compose), they will not be equipped with a prefix operator. This will help to distinguish levels from quantities in expressions. The quantity [+12 h +15 min] differs from the point of time 12:15 (no leading '+'). The length '+3 m' differs from the altitude '3 m'.

The mechanism of creating level values can be demonstrated by $0\,\mathrm{m}\ {+3}\,\mathrm{m}\ = 3\,\mathrm{m}$, or simplified as: $0\ {+3} = 3$.[1] Here 0 m is the reference level. This and the quantity +3 m form an anchor—agent relation.

> This kind of mathematics is established. It can be found in Lie algebra, topologies, and so on. It is an example of 'group action'.

Some significant features of levels are ...

- Their "double reference" feature;
- Their lack of operation. And accordingly ...
- Their confinement to work as 'anchors', not as agents.
- They can indeed be compared! (This being the only "operation" *between* levels.)

> A remark on the use of '※' and '♀': The ※ symbol is used for comparisons between agents, producing an agent of the same kind as an answer. (It may also be used as a general comparison symbol.) The ♀ symbol denote comparisons between "anchors", producing a corresponding agent. Moreover, this agent is intended to show the "numeracy" of the reference anchor. (This rules out using ♀ for level comparison. There ※ is used.) This restriction makes ♀ more useful, e.g. $+5x^3\ {-15x}$ ♀ $+x^2\ {-3} = \cdot 5x$. Such an expression can be quite accessible to learners, compared to a quotient between polynomials.

To infer how comparison between levels works, we set off by taking a look at how quantity comparison could be performed:

The comparison +18 ♀ +3 = ·6 can be outlined as an "extraction" of measures (i.e. scalars) from the quantities, followed by a comparison between scalars. As $+18 = +1 \cdot 18$ and $+3 = +1 \cdot 3$, where $\cdot 18$ and $\cdot 3$ are measures, we substitute the comparison between quantities for its scalar equivalent: ·18 ※ ·3 = ·18 ∾·3 = ·18 /3 = ·6.

Now we apply these patterns on 'levels'. To perform the comparison 7 ※ 5, we begin by extracting the enclosed "inner" quantities. As $7 = 0\ {+7}$ and $5 = 0\ {+5}$, the quantities will be +7 and +5. Then we have that +7 ※ +5 = +7 ∾+5 = +7 −5 = = +2, telling that 7 lies 2 (unit) steps from 5, in the positive direction.
From comparisons of this kind, level values can be *ordered.* Nevertheless, levels have no inherent "size" attribute. The position 7 is not "greater" than the position denoted by 5 on the number line, it only comes "later" than 5 in the order.

[1] ... by regarding the unit as abstract and as such, implicit.

The comparison of levels follows the usual pattern, that an agent tells how to transform (translate) the $\mathcal{R}$ anchor into the $\mathcal{Y}$ anchor. From $5 +2 = 7$ we see that $7 ※ 5 = +2$. This can be demonstrated on a number line, where levels are coordinates and quantities are depicted as arrows between coordinates.

> We can see that levels mathematically work very different from quantities. This is established knowledge used in e.g. topology. Nothing is told about those 'level' features in compulsory school education.[①] Textbook authors even compromise how levels work, e.g. by multiplying temperatures with scalars. Typically, negative numbers are exemplified by "levels" like temperature, altitude and coordinate points. As a next step, it is demonstrated how to add, subtract, multiply and divide negative numbers, in spite of the fact that a pair of real-world "levels" cannot be subject to a binary operation. So what do the pupils learn from those examples? Are they of any help? The only operation a level submits to, is a "translation" to another level by means of a quantity.

The concept of level elements complies with the structure given by the $\mathcal{RAY}$ model. The anchor and agent concepts help sorting out the component roles.

Powers

I will not go into details about powers. Only a short survey of how the $\mathcal{RAY}$ model and anchor/agent concepts are applied will follow.

- The power base works as an anchor, and is a scalar.
- The exponent is an agent of its own kind. The '$\hat{}$' (hat) is used as an operator.
- Following the pattern of $5 \cdot 3 = +5 \cdot 3 = +5 +5 +5 = +15$, we have that $5^3 = \cdot 5 \;\hat{}3 = \cdot 5 \cdot 5 \cdot 5 = \cdot 125$.

> You may say that scalars "repeat" quantities and exponents "repeat" scalars. This must not be confused with "repeated addition" or "repeated multiplication". For those concepts, what is being repeated is frequently mistaken to be an operation, not an element.

- The comparison $\cdot 125 ♀ \cdot 5$ yields $\hat{}3$, as is obvious from $\cdot 5 \xrightarrow{\hat{}3} \cdot 125$. This corresponds to computing the "five logarithm of 125". (The comparison $\cdot 125 ※ \cdot 5$ yields $\cdot 25$, N.B.)
- The opposite operation to '$\hat{}$' is denoted '$''$' and returns a root. Accordingly, if $\cdot x \;\hat{}3 = \cdot 125$, then $\cdot x = \cdot 125 \,\backsim\!\hat{}3 = \cdot 125 \,''3 = \cdot 5$, is a solution via "back-ward reckoning". Mind that $\mathcal{R}$ in $\mathcal{RAY}$ stands for 'root' or 'reference'. (Regarding $\mathcal{R}$ a "root" in the $\mathcal{RAY}$ interpretation of a power makes sense.)
- The associative law is applied as $[\cdot 2 \;\hat{}3] \;\hat{}4 = \cdot 2 [\hat{}3 \;\hat{}4] = \cdot 2 \;\hat{}12$, where the composing of exponents by multiplication, easily can be derived.
- One form of the distributive law applies according to $[\cdot 2 \cdot 3] \;\hat{}4 = \cdot 2 \;\hat{}4 \cdot 3 \;\hat{}4$ in the very exact parallel to $[+2 +3] \cdot 4 = +2 \cdot 4 +3 \cdot 4$.

① The "transfer" of university level mathematical insights to compulsory school, seems vanishingly small.

The other form of the distributive law looks like $\cdot 2\ \hat{}\,(+3{+}4) = \cdot 2\ \hat{}\,(+3)\ \cdot 2\ \hat{}\,(+4)$ or simplified like $\cdot 2^{+3+4} = \cdot 2^{+3}\ \cdot 2^{+4}$. Regrettably, treating the theory behind this is an essay on its own.

- A change of the "inner sign" of the exponent, as between $\hat{}\,(-2)$ and $\hat{}\,(+2)$ is coupled with a sign change for the base (anchor), that is

$$\cdot 2\ \hat{}\,(-3) = \backsim\!\cdot 2\ \hat{}\,(\backsim\!-3) = /2\ \hat{}\,(+3) = /2\ /2\ /2 = /8.$$

The transformation $\cdot 2\ \hat{}\,(-3) = /2\ \hat{}\,(+3)$ can read "3 less '$\cdot 2$' equals 3 more '$/2$' ". Seen as agents working on $\cdot 2\ \cdot 2\ \cdot 2\ \cdot 2\ \cdot 2$ this corresponds to

$$\cdot 2\ \cdot 2\ \cancel{\cdot 2}\,\cancel{\cdot 2}\,\cancel{\cdot 2} = \cdot 2\ \cdot 2\ \cdot 2\ \cdot 2\ \cdot 2\ /2\ /2\ /2.$$

This is perfectly paralleled by

$$+2\ \cdot(-3) = \backsim\!+2\ \cdot(\backsim\!-3) = -2\ \cdot(+3) = -2\ -2\ -2 = -6,$$

saying "3 less '$+2$' equals 3 more '-2' ".

Perhaps this gives you a hint on the issue of what I call "inner quantities" in scalars and exponents, along with their contribution to simplified "sign rules". Those are not to be treated upon, more than this in this paper. Here the focus is on real-world structure. Anyhow, my little "hint" on sign rules might have put you on the track of seeing that the interpretation of a negative number depends on what "role" the number plays – does it represent a level, quantity, scalar or exponents?

On fractions

There's much to be said about fractions. Only a fraction will be treated here. One thing is that they form rational numbers, defined as equivalence classes (of the kind, earlier mentioned). Another is that positive rationals form a group under multiplication. That hints at that rationals can be associated with scalars.

The main point has already been touched on. A general comparison of type $+a$ ♀ $+b$, produces the composed scalar $[\cdot a\ /b]$ as an answer.
From $+1\ [\cdot a\ /b] = +a\ /b = +{}^{a}\!/_{b} = +1\ \cdot {}^{a}\!/_{b}$, it follows that $\cdot {}^{a}\!/_{b} \equiv [\cdot a\ /b]$.[1] This interpretation of a fraction, as a combined multiplication and division, is sometimes referred to as "the operator model". In a misguided ambition to have but *one* model for fractions, the operator model is questioned. On the contrary, it is the basic notion of fractions. This is why ...

A scalar can always work as a measure. Just as $\cdot 5$ in $+5\,\mathrm{s} = [+1\,\mathrm{s}]\ \cdot 5$ is the measure of five seconds, so is $\cdot {}^{a}\!/_{b}$ the measure of $+{}^{a}\!/_{b}$. The operator model and scalar fraction is indeed the basis, as the scalar works as a measure that *creates* the quantitative fractions.

Many models for fractions, use *quantities* in the shape of rectangular or circular (pie) areas, or stretches with length. To demonstrate ¾ of a rectangle, it is divided in four parts, whereof three may be shadowed (or otherwise marked). What is neglected, is that a *mathematical* creation of ¾ of an area is not performed by means of a ruler and a pencil. It is done by the two operations $/4$ and $\cdot 3$. They are applied on a area being a quantity. Here the language used about fractions

[1] The equality $\cdot {}^{a}\!/_{b} = [/b\ \cdot a]$ can as well be stated as a definition of a scalar fraction. Observe, that the unit $+1$ works as a quantitative anchor to the scalar agent, being the measure.

set a trap. After having divided 'the whole' (e.g. a rectangle area) by four (i.e. applied $/4$ on $+1$ leaving $+\frac{1}{4}$) we name the part a 'quarter'. Then we take 3 of them: $+\frac{1}{4} +\frac{1}{4} +\frac{1}{4} = +\frac{1}{4} \cdot 3 = +\frac{3}{4}$.②

The trap is, that if understanding fractions makes use of quarters, fifths and other parts as a basis, attention is drawn away from the role of the denominator to work as a divisor. It reduces into a "nomen" (name) for the part. Only the multiplicator role of the numerator is properly recognized. Parts will be created in mind or by practical arrangements, not by division. The parts will be perceivied as 'unit fractions', that is: *quantities*. (The use of leading +'s, like in $+\frac{1}{2}$, $+\frac{1}{3}$, $+\frac{1}{4}$ and so on, would expose this property.)

With fractions understood this way, it will be hard to construe the meaning of $\frac{1}{3} \cdot \frac{1}{4}$ as you cannot multiply two pieces of a pie, to use a metaphor. When the pupil is asked to write an expression showing "three quarters of a", the trap definitely shuts.①

Teachers are told in their education, that the 'operator model' is not central and has the draw-back of not fitting in with other models. (Those models seem to be classified after what kind of real-world object the fraction is applied on.② To me, that is not mathematics.)

The $\mathcal{RAY}$ model provides the structure needed. The scalar $\cdot a/b$ is to be understood as $[/b \cdot a]$ or $[\cdot a \,/b]$. It shows the relation between quantities. It is construed with the two-step method as in $+5 \xrightarrow{/5} +1 \xrightarrow{\cdot 2} +2$ implying that $+2 ♀ +5 = = [/5 \cdot 2] = \cdot 2/5$. The method and reasoning is, due to my experience, easy to convey to pupils, with a helping hand from hands-on materials. You can apply a (scalar) fraction on "a whole", that is $+1$, regardless of the unit will be a 'pie', 'square', 'glass of water', etc. It makes no difference to apply a fraction on a number of objects, i.e. $+n$. In either case the application of a scalar on a quantity yields a quantity: $+1 \cdot 2/5 = +2/5$ or, if $+n = +35$: $+35 \cdot 2/5 = +14$.③

When you tell apart representations for quantities and scalars, you make way for attending the parallel between (concrete) division by a knife, and (mathematical) division by a number. (The quantitative "pie anchor" is, so to say, submitted to the operation of a surgeons knife or a mathematicians divisor.)

The interpretation of a/b as $[+1/b] \cdot a$ manifest the widespread view of the denominator as a kind of "unit" (telling the "size") and the numerator as the "number" (i.e. works as a multiplier). As mentioned before, this leaves out the mere *creation* of $+1/b$ as depicted of $+1\,[/b \cdot a] = [+1\,/b] \cdot a = +1/b \cdot a$. Indeed, this initial step is crucial for getting a firm grip on how to manage fractions.

② Conventionally this is written $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3 \cdot \frac{1}{4} = \frac{3}{4}$. Here nothing prevents the quantity property of the (unit) fraction from being overlooked. Mind: things being added are quantities.

① The division by the denominator will not likely be an active part of the pupils model of a fraction. How to make four parts of "a" by means of a ruler or by "pie dividing" techniques?

② The use of a square to model fraction multiplication is considered different from using a rectangle! Somehow someone somewhere seems to have missed the point.

③ This extensive use of '+' signs (and others) might seem to clutter up the expressions. The intention of using them is only to make the role of numbers in an expression explicit, and to facilitate reasonings about numbers and their correspondences in reality. Nothing prevents the experienced user of mathematics to leave out signs where they seem to be of minor use. The more complex expressions, the more the need for simplification.

Composition of scalars

Understanding calculations with fractions is but a part of understanding scalar expressions in common. How to make a pupil understand a rearrangment like the following?

$$\frac{1}{3} \cdot 5/4 \cdot \frac{6}{5} = \frac{5}{5} \cdot \frac{6}{3 \cdot 4}$$

> In 2002 Wiggo Kilborn presented (due to a study) that as many as 42 % of the pupils in form 9, couldn't calculate $\frac{6}{5}/3$! If the pupils had been able to regroup the scalars as into $\frac{6}{3}/5$, there would not have been a problem.

To facilitate rearrangements of scalar expressions, one needs to identify what *affect* each number has *on the expression as a whole*, with respect to its operation. As of the expression above, the *roles* the numbers posess can be demonstrated by $\cdot 1 \,/3 \,\cdot 5 \,/4 \,\cdot 6 \,/5$. (Leading signs, N.B.) After having left out the redundant '$\cdot 1$' and done some regrouping, this equals $[\cdot 5 \,/5][\cdot 6 \,/3] \,/4 = \cdot 1 \,\cdot 2 \,/4 = /2$.

In parallel with the concepts 'plus term' and 'minus term', that are useful tools to classify term agents, we need similar concepts for scalars. I propose we name the factor '$\cdot a$' and its converse '$/a$' a 'profactor' and an 'antifactor', respectively.①

> Other terms would be possible, for instance 'sizefactor' and 'partfactor'. The scalar '$\cdot m$' can be regarded a "measure" of something, as scalars often are. The signs themselves can read 'times/part', 'pro/anti' or some other short forms. Cumbersome wordings like 'multiplied with' and 'divided by' are not likely to be used in practise when long expression are read aloud. To avoid "baby talk" in mathematics, the short forms should be settled on.

Here is a suggestion on how to demonstrate the scalar "roles" in expressions. For "one-liners" like $\cdot 3 \,\cdot 4 \,/5 \,/6 \,\cdot 7$ a consistent use of prefix interpretation of signs tells the role of each number. For "two-liners" in quotients the interpretation is straightforward: Above the fraction bar (in the numerator), the affect on "the expression as a whole" is unchanged. An $\cdot a$ will even in a wider scope (outside the numerator) work as a profactor and $/a$ as an antifactor. Below the fraction bar (i.e. in the denominator) the role of a factor will be reversed. Here the fraction bar (a vinculum) takes on the role of grouping factors and submit those below to an inversion as by the sign '∾'.

$$\frac{\cdot 6 \,\cdot 7 \,/10}{\cdot 7 \,/5 \,\cdot 3} = [\cdot 6 \,\cdot 7 \,/10] \backsim [\cdot 7 \,/5 \,\cdot 3] = [\cdot 6 \,\cdot 7 \,/10][/7 \,\cdot 5 \,/3]$$

> Naturally, a vinculum can be regarded as a "token" for regular division and, concurrently, may imply some sort of inverting, following less transparent rules. A subsequent, more transparent and immediate reasoning follows. This can, in its turn, be used to prove how the division operation can be used to invert factors.
>
> From a theoretical standpoint, the inverting property of division is quite obvious, as division is derived from the basic concept of inversion. (The same goes for subtraction.) In school mathematics, where division as a concept is brought up first, the other-way-round inference of the inverting property becomes a difficulty later on. (This inverting property is introduced for managing comparisons and backward reckoning.)

① Naming e.g. $/2$ a 'factor' is motivated by that you cannot tell it apart from $\cdot 0.5$ when you deal with coherent elements, composed of an operation and a number: $/2 \equiv [\cdot 1 \,/2] \equiv \cdot 0.5$.

Instead, the *idea* behind why reversing (of operations) is needed in the first place, should be brought into daylight using the $\backsim$ operator. The use of ordinary operators for this purpose should be introduced only as a second step.

In effect, the role of a factor above or below the vinculum can be derived by means of comparison. Any pupil can learn that a comparison with the unit quantity $+1$ works according to $+a \,♀\, +1 = \cdot a$ since $+1 \xrightarrow{\cdot a} +a$. Now use (what I call) the "trivial fraction" $+{}^1\!/\!_1 = +1$. Make it "change" by inserting factors in the numerator and/or denominator. Then determine the overall "change" by comparing the result with the origin $+1$.

For example does an $/5$ inserted in the denominator of $+\frac{1}{1}$ yield ...

$$+\frac{1}{1\,/5} = +\frac{1}{0.2} = +5$$

Then $+5 \,♀\, +1 = \cdot 5$ tells that the insertion of $/5$ made the expression as a whole 5 times larger. The general conlusion will be that an antifactor inserted in the denominator works as a profactor. The pattern of that factors in the numerator keep their roles but have them reversed in the denominator is simple to learn and derive.

With those insights the pupil can perceive $\frac{6}{5}/3$ as

$$\cdot 6\,/5\,/3 = [\cdot 6\,/3]\,/5 = \cdot\frac{6}{3}\,/5 = \cdot 2\,/5 = \cdot{}^2\!/\!_5$$

The problem of "rearrangement" melts down to identifying the factor roles in the one expression, and write the other with their roles maintained. The 'neutral factors' $\cdot 1$ and $/1$ can be left out or inserted as one pleases. Exercises in rewriting make room for the pupils creativity, and ease the way to algebra.

Measurement division and proportionality

The correspondence between the functionality of a quotient and a comparison is obvious:

$$\mathcal{Y} ※ \mathcal{R} = \mathcal{Y} \backsim \mathcal{R} = \frac{\mathcal{Y}}{\mathcal{R}}$$

As earlier mentioned, the use of a quotient for comparison is often referred to as 'measurement division'. That is an old notion. The ancient Greeks said that one quantity *measures* another if it (due to our speaking:) divides into it. Two thousand years later John Wallis (Oxford prof.) stated that *different* kinds of quantities may be compared by means of a quotient, *provided* the ratio was between "pure numbers" without any units or other quantity features.

The underlying reason is that a combination of scalar measures ("pure numbers") in a formula shall reflect how one quantity depend on others. E.g. if 'velocity' of size $+1$ is defined as that of a $+1\,\mathrm{m}$ move in $+1\,\mathrm{s}$, then the measure $\cdot v$ of velocity will be directly proportional to the strech measure $\cdot s$ and reciprocally proportional to the time measure $\cdot t$. Accordingly the formula $v = {}^s\!/\!_t$ is all about measures, not quantities. The operation "strech divided by time" remains to be performed.

Textbook authors often emphasize on the idea, that comparison division is between quantities, by spelling out the units in the quotient as in

$$\frac{10\,\mathrm{m}}{10\,\mathrm{cm}} = 100 \quad \text{or} \quad \frac{35\,\mathrm{km}}{5\,\mathrm{h}} = 7\,\mathrm{km/h}$$

Nevertheless, the pupil gains from being aware of that in those expressions, the vinculum only is a symbol for comparison. That is ...

$$\frac{+35\,\mathrm{km}}{+5\,\mathrm{h}} = +35\,\mathrm{km} \text{ ♀ } +5\,\mathrm{h} = \cdot 35 \text{ ※ } \cdot 5 = \cdot 35 \text{ ∾} \cdot 5 = \cdot 35\,/5 = \cdot\frac{35}{5} = \cdot 7$$

Now $\cdot 7$ is the *measure* of the velocity!

The expression '$\cdot 35$ ※ $\cdot 5$' above does what Wallis told us to do: it compares "pure numbers".[1] Behind all this lies that the relation between the quantities is the one of proportionality. The relation itself constitutes a quantity – a velocity. Its measure equals that of the strech traversed during the time unit $(+1)$ measure $(\cdot 1)$. When velocity is constant, strech and time are proportional, implying they change in step. In our example the factor $/5$ yields unit time: $+5\,\mathrm{h} \xrightarrow{/5} +1\,\mathrm{h}$. Applied likewise on the strech, it yields $+35\,\mathrm{km} \xrightarrow{/5} +7\,\mathrm{km}$. Now the strech measure $\cdot 7$ is the velocity measure, per definition.

The Wallis method is a short cut. Cumbersome proportionality reasonings are left out and he goes directly for the measure that may be determined from the strech and time measures in a quotient. It can be questioned wether pupils shall learn this short cut without understanding the underlying proportionality reasoning. I have tried and found the "long way round" to be a short cut to comprehension.

The use of a quotient for direct proportionality actually reflects that two measured phenomena change the same way.

Example: If the strech $12\,\mathrm{m}$ is proportional to the time $5\,\mathrm{s}$ (constant velocity), then a 3 times longer strech is accompanied by a 3 times longer time. If the quantites are arranged in a quotient, those changes will leave the quotient unchanged: $\frac{12}{5} = \frac{12\cdot 3}{5\cdot 3}$ or, fully featured:

$$\frac{+12\,\mathrm{m}}{+5\,\mathrm{s}} = \frac{+12\,\mathrm{m}\cdot 3}{+5\,\mathrm{s}\ \cdot 3} = \frac{+36\,\mathrm{m}}{+15\,\mathrm{s}}$$

Written with quantities, the quotient gains from being interpreted to show a *relation*, rather than a division. A quotient between plain measures is another cup of tea. It's extended like:

$$\frac{12}{5} = \cdot 12\,/5 = \cdot 12[\cdot 3\,/3]\,/5 = [\cdot 12\ \cdot 3][/5\,/3] = \frac{12\cdot 3}{5\cdot 3}$$

In effect: **we multiply and divide by the same number**, which explains why the quotient remains unchanged.[2]

Measures for reciprocal proporitionality are usually set up as a product. Here, too, we multiply and divide by the same number. It looks like:

$$a\cdot b = \cdot a\ \cdot b = \cdot a[\cdot c\,/c]\ \cdot b = [\cdot a\ \cdot c][\cdot b\,/c] = ac\cdot\frac{b}{c}$$

It becomes evident, why a product is suitable to show reciprocal proportionality.

[1] It is, but of course, impossible to figure out "how many times 5 h divides into 35 km"!
[2] Understanding connections, thrills pupils.

To conclude

The $\mathcal{RAY}$ and anchor—agent models combined with bound operators comprise what I call "Change and Comparison Mathematics" – CCM. The concepts lead considerably beyond what is outlined here.

> Among other things: Proportionality and negative numbers become straightforward matters. The angle concept can be settled on. Ways to interpret and generalize complex numbers are pointed at.

Basically, the concepts in CCM are aimed at making it more simple to understand basic arithmetical structures. This is the CCM raison d'être. There's no need for alternative theories like this one, just to make out calculations. What justifies its existence is the extent to which it clarifies on diverse issues, and does so in a simple, consistent way, that readily lends itself to be demonstrated by hands-on materials. This is of major importance, as pupils frequently are able to perceive how to handle mathematics in practical situations. Almost equally frequent, they are less able to transfer their insights from practice over to written expressions and general structures.

Many features of CCM can be found elsewhere, in a slightly different outfit. There, those features are aimed at solving complex problems and, accordingly, not well adapted to be conveyed to children. In Russia, the Davydovian way of mathematical instruction in form 1–3, have several outfits in common with CCM. This, too, talks in favour of CCM.

CCM lays a foundation, that complies with advanced modern theories, but avoids all their complexities. It is meant to provide a tool to bridge the gap between abstract symbol expressions and real-world mathematical structures. To my experience, it does so surprisingly well. Will this be the experience of others, too?