The Taxicab number 1729

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The story of Hardy visiting Ramanujam at his sick-bed and referring to the boring number - 1729, on the cab that brought him there, only to be contradicted by the patient who informed him that it was the smallest integer that could be written as the sum of two cubes in two different ways ($12^3 + 1^3 = 10^3 + 9^3$) is of course well-known. Hardy was taken aback and asked about the corresponding number for fourth powers, only to be told that Ramanujam did not know but suspected it was very high.

The moral of the story is of course to indicate the amazing familiarity Ramanujam had with individual numbers. As a mathematical problem, Hardy probably did not think it was many cuts above the kind of trivial facts that he ridiculed in his Apology and which formed the back-bone of such publications as that of Rose on recreational mathematics (later to be revised by Coxeter). Trivial or not, this is the kind of thing that can intrigue the young individual, and most of us probably came across the story early in life.

It is very unlikely that Ramanujam ever systematically looked for such numbers, his knowledge of the fact was probably just a fortuitous spin-off from other considerations, and even if he had done so, it is unlikely that he would have gotten far. Nowadays with the advent of high-powered computers anyone with some minimal skill of programming can make a search in a few seconds. As a curiosity I list all the primitive solutions up to one million. By a primitive solution I mean one in which the numbers involved have no common factor, if they do, they arise in a trivial way from a solution in which there is no common factor but the trivial one.

$1729 = 12^3 + 1^3 = 10^3 + 9^3$	$4104 = 16^3 + 2^3 = 15^3 + 9^3$	$20683 = 27^3 + 10^3 = 24^3 + 19^3$
$39312 = 34^3 + 2^3 = 33^3 + 15^3$	$40033 = 34^3 + 9^3 = 33^3 + 16^3$	$64232 = 39^3 + 17^3 = 36^3 + 26^3$
$65728 = 40^3 + 12^3 = 33^3 + 31^3$	$134379 = 51^3 + 12^3 = 43^3 + 38^3$	$149389 = 53^3 + 8^3 = 50^3 + 29^3$
$171288 = 55^3 + 17^3 = 54^3 + 24^3$	$195841 = 58^3 + 9^3 = 57^3 + 22^3$	$216027 = 60^3 + 3^3 = 59^3 + 22^3$
$327763 = 67^3 + 30^3 = 58^3 + 51^3$	$402597 = 69^3 + 42^3 = 61^3 + 56^3$	$439101 = 76^3 + 5^3 = 69^3 + 48^3$
$443889 = 76^3 + 17^3 = 73^3 + 38^3$	$515375 = 80^3 + 15^3 = 71^3 + 54^3$	$684019 = 82^3 + 51^3 = 75^3 + 64^3$
$704977 = 89^3 + 2^3 = 86^3 + 41^3$	$805688 = 93^3 + 11^3 = 92^3 + 30^3$	$842751 = 94^3 + 23^3 = 84^3 + 63^3$
$920673 = 97^3 + 20^3 = 96^3 + 33^3$	$955016 = 98^3 + 24^3 = 89^3 + 63^3$	$984067 = 98^3 + 35^3 = 92^3 + 59^3$
$994688 = 99^3 + 29^3 = 92^3 + 60^3$		

Incidentally there is no number below a million that can be written as a cube in three different ways.

While we are at it, we can also try to look for numbers that can be written as the sum of two fourth powers in two different ways. None is found below a million, and indeed Ramanujam was right. Yet a wider, much more time-consuming search reveals

$239^4 + 7^4 = 227^4 + 157^4$	$158^4 + 59^4 = 134^4 + 133^4$
$542^4 + 103^4 = 514^4 + 359^4$	$292^4 + 193^4 = 257^4 + 256^4$
$631^4 + 222^4 = 558^4 + 503^4$	$502^4 + 271^4 = 497^4 + 298^4$

The curious thing is that the problem of representing numbers that can be written as the sum of two cubes in two different ways can actually be completely solved, at least in the sense that a parametric solution can be written down. This goes back to the fact discovered early in the 19th century that the cubic surface in P^3 has twenty seven lines, which can be used to show that it is rational, i.e. that it has a rational parametrization. In fact it can be parameterized by four cubic ternary forms. The geometric construction that lies behind it can also be used to show that this can be done over various fields.

As a warmup we will consider the far more elementary problem of when a number can be written as a sum of two squares in two different ways.

$X^2 + Y^2 = Z^2 + W^2$

This is indeed a far more elementary problem. In fact we can rewrite it as $X^2 - Z^2 = W^2 - Y^2$ or $(X+Z)(X-Z) = (W+Y)(W-Y)$. This gives a clear clue how to generate non-trivial solutions. Take a number N which has two different factorizations $u_1v_1 = u_2v_2$. If they are of the same parity we can solve for integral values $X + Z = u1, X - Z = v1, W + Y = u_2, W - Y = v_2$ leading to

$$\frac{(u_1+v_1)^2}{4} + \frac{(u_2-v_2)^2}{4} = \frac{(u_1-v_1)^2}{4} + \frac{(u_2+v_2)^2}{4}$$

The simplest case with $u_1 = 5, v_1 = 3, u_2 = 15.v_2 = 1$ gives rise to the identity $7^2 + 4^2 = 1^2 + 8^2 = 65$ in fact this is the smallest number which can be written as a sum of two non-zero squares in two different ways. (Otherwise we have of course $5^2 = 4^2 + 3^2$).

We can think of the parameter space as given by (u_1, v_1, u_2, v_2) subject to the condition $u_1v_1 = u_2v_2$. This condition is given by another quadric in P^3, so in fact it is rather tautological. The difference being that we are more adept at finding two numbers that multiply to the same product, then finding two different differences of squares being equal, although the two things are equivalent.

We can also use geometry to get a parametrization. The quadric $X^2 + Y^2 = Z^2 + W^2$ has an obvious solution $P = (1, 0, 1, 0)$. As our parameter space we consider points on the hyperplane $X = 0$ i.e. points $(0, u, v, w)$. The lines joining them to P have parametric representation

$$\lambda(1,0,1,0) + \mu(0,u,v,w) = (\lambda, \mu u, \lambda + \mu v, \mu w)$$

inserting into the quadric gives the binary quadric

$$\lambda^2 + \mu^2 u^3 - (\lambda + \mu v)^2 - \mu^2 w^2$$

which simplifies to

$$\mu(-\lambda(2v) + \mu(u^2 - v^2 - w^2))$$

This will of course vanish for $\mu = 0$ corresponding to the point P. The residual intersection with the quadric will be given when we set the other factor to be zero, i.e.

$$\begin{aligned} \lambda &= u^2 - v^2 - w^2 \\ \mu &= 2v \end{aligned}$$

Plugging that in in the parametric representation of the line we get the parametrization of the quadric given by

$$(u^2 - v^2 - w^2, 2uv, u^2 + v^2 - w^2, 2vw)$$

and the reader can easily verify the polynomial identity

$$(u^2 - v^2 - w^2)^2 + (2uv)^2 = (u^2 + v^2 - w^2)^2 + (2vw)^2$$

Numbers which are the sum of two squares or two cubes

Given a number n one can verify that it is the sum of two squares without having to find them. It is well-known that this happens iff every prime-factor of type $4k+3$ has even multiplicity in the prime-decomposition of n. The number of different representations is given by 2^{k-1} where k is the number of prime-factors of type $4k+1$. If there are none, the number of representations is 1.

The key idea is that a prime p of type $4k+1$ in $\mathbb{Z}$ splits into two non-associated primes $a \pm ib$ in the ring $\mathbb{Z}[i]$ of Gaussian integers, while $2 = (1+i)^2$ becomes a square. For such primes p we have $p = Nm(a+ib) = a^2 + b^2$ and the representation is unique (up to trivial modifications).

In particular if there are many prime-factors of the right type, there will be many representations as a sum of two squares. E.g. we have for $1105 = 5 \times 13 \times 17$ we have

$$9^2 + 32^2 = 23^3 + 24^2 = 33^2 + 4^2 = 31^2 + 12^2$$

This comes from $5 = (2+i)(2-i), 13 = (3+2i)(3-2i), 17 = (4+i)(4-i)$, allowing us to pick up four different pairs of conjugate Gaussian numbers which multiply to 1105.

Now to find the representation of a prime p of the right type as a sum of square can of course be done by trial and error for small primes, or by computer searches for somewhat larger primes. One way of speeding up the process is to find a number n such that $p|(n^2+1)$ which is equivalent to solving the congruence equation $x^2 = -1(p)$.

This can be done by taking powers of elements modulo p, say if $p = 37$ then consider $2, 4, 8, 16, 32, 27, 17, 34, 31, 25, 13, 26, 15, 30, 23, 9, 18, 36 = -1(37)$ formed by doubling each consequetive number, which can be done in your head. 36 is the 18th power of 2 (incidentally this shows that 2 is a primitive root modulo 37), thus we look for the 9th power which is 31. Hence $37|1+31^2$ which might not have been so easy to find out by trial and error.

The next step is to do the Euclidean algorithm on $n+i$ and p, which works for the Gaussian integers.

We have that $13|5^2+1$. Consider $\frac{13}{5+i}=\frac{13(5-i)}{26}=\frac{5-i}{2}=2+\frac{1-i}{2}$ thus we can write $13=2(5+i)+(3-2i)$. Similarly $\frac{5+i}{3-2i}=\frac{(5+i)(3+2i)}{13}=\frac{13+13i}{13}=1+i$ hence $5+i=(1+i)(3-2i)$ and we get that $13=(3-2i)(3+2i)=3^2+2^2$

Note that a^2+b^2 is simply the norm of a Gaussian integer, and an example of a binary form. A similar binary form is given by a^2-ab+b^2 which are the norms of the integers $\mathbb{Z}[\rho]$ where ρ is a primitive cube root of unity. The situation is completely analogous to the more well-known Gaussian case. A prime is the norm of what I would prefer to call the Fermat case[0] iff it is of form $6k+1$ or 3, in the latter case it is associated to a square. Whether an arbitrary number can be so represented, you need to find its prime-decomposition, and ascertain that every prime-factor of type $6k-1$ (or 2) occurs with even multiplicity. To find those representations for a suitable prime p you need to find a non-trivial solution to the congruence equation $x^3=-1$ which can be done as before. This means finding a number n such that $p|(n^2-n+1)$ and then to use the Euclidean algorithm to find the greatest common divisor of p and $n+\rho$ in $\mathbb{Z}[\rho]$.

37 is a suitable prime in this context as well. Having done most of the work already, we only need to find the 6th power of two ($6=18/3$ but 12 would do equally well) which is 27. We then only need to do the Euclidean algorithm on $27+\rho$ (or $26+\rho$ if you prefer) to end up with the splitting of 37 in $\mathbb{Z}[\rho]$. Or just guess and try. $7+3\rho$ works fine as $7^2-21+3^2=37$.

Now because $\mathbb{Z}[\rho]$ has so many units, in fact six given by $\pm1,\pm\rho,\pm(1+\rho)$ forming a regular hexagon in the complex plane, there are many associates to $a+b\rho$. In fact the following twelve numbers are either associated or associated to the conjugate of a given $a+b\rho$ and thus have the same norm. (The twelve numbers are distinct except in the case of the norm being equal to 3). Note that $\overline{a+b\rho}=(a-b)-b\rho$ as $\rho^2=-(1+\rho)$.

(a,b)	(a-b,-b)
(-a,-b)	(b-a,b)
(-b,a-b)	(-a,b-a)
(b,b-a)	(a,a-b)
(a-b,a)	(-b,-a)
(b-a,-a)	(b,a)

In particular for future reference if we add the two numbers, we get six different possibilities

$$\pm(a+b),\pm(2b-a),\pm(2a-b)$$

Now, just as in the Gaussian case a number with many factors will many have many different representations as a^2-ab+b^2.

Now if a given number n is the sum of two cubes $n=x^3+y^3$ we can factor it into three factors in $\mathbb{Z}[\rho]$. Namely $x^3+y^3=(x+y)(x+\rho y)(x+\rho^2y)=(x+y)(x^2-xy+y^2)$

[0]The ring of integers to the elliptic curve $y^2=x^3+ax$ (associated to the Lemniscate) is the Gaussian integers, while the ring corresponding to the Fermat cubic $y^2=x^3+a$ is the ring $\mathbb{Z}[\rho]$.

of which the last two are combined in $\mathbb{Z}$. Thus if n should be the sum of two cubes, we should be able to factor $n = fN$ where N is a norm, and $f = x + y$ for some x, y such that $Nm(x + \rho y) = N$. Now due to the identity

$$\frac{(x+y)^2 + 3(x-y)^2}{4} = x^2 - xy + y^2$$

we see that $f \leq 2\sqrt{N}$, which gives some restriction on the possible N.

We have that $1729 = 7 \times 13 \times 19$. Furthermore $19 = Nm(5+3\rho), 13 = Nm(4+\rho), 7 = Nm(3 + \rho)$ The possible N are $19 \times 7, 13 \times 7, 19 \times 13$. For each of those products we can find the essentially different representations (modulo those that are generated by the scheme above). We can represent them in a table.

19×7	$(5+3\rho)(3+\rho) = 12 + 11\rho$	$(5+3\rho)(2-\rho) = 13 + 4\rho$
$\lvert a+b \rvert$	23	17
$\lvert 2a-b \rvert$	**13**	22
$\lvert 2b-a \rvert$	10	5
13×7	$(4+\rho)(3+\rho) = 11 + 6\rho$	$(4+\rho)(2-\rho) = 9 - \rho$
$\lvert a+b \rvert$	17	8
$\lvert 2a-b \rvert$	16	**19**
$\lvert 2b-a \rvert$	1	11
19×13	$(5+3\rho)(4+\rho) = 17 + 14\rho$	$(5+3\rho)(3-\rho) = 18 + 7\rho$
$\lvert a+b \rvert$	31	25
$\lvert 2a-b \rvert$	10	29
$\lvert 2b-a \rvert$	11	4

Due to the two high-lighted values, we see that 1729 can indeed be written as a sum of two cubes in two different ways.

A similar, if somewhat more involved, analysis can be made for the case of 4104.

Frequency of sum of two cubes

How common is it that a number is the sum of two cubes? It obviously depends on the size of the number, the larger, the less likely. Let us start to ask what about the probability of a number being a square. If the square is about size N, the next square is about $N + 2\sqrt{N}$, thus with a gap of $2\sqrt{N}$. The probability of being a square would then be about $1/2\sqrt{N}$. What about being the sum of two squares? We consider the numbers $N, N-1, N-4, N-9 \ldots N - (\sqrt{(N/2)})^2$ and the probability that each of them is a square. This is paramount to computing the product

$$1 - (1 - x_1)(1 - x_2)\ldots(1 - x_n) = \sum_i x_i - \sum_{i<j} x_i x_j + \sum_{i<j<k} x_i x_j x_k \ldots$$

for some large n. If the first sum is small, we can ignore the other sums. To get an idea of the first sum we consider the integral as a good approximation of the first sum.

$$\int_0^{\sqrt{N}/2} \frac{dx}{2\sqrt{N - x^2}}$$

By making the obvious substitution $y = \frac{x}{\sqrt{N}}$ we reduce to an integral we can actually evaluate (using arcsine).

$$\frac{1}{2}\int_0^{1/\sqrt{2}} \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{8}$$

This is big, which means that we cannot ignore the higher order sums. In fact to compute the frequency of such numbers is a very delicate matter as shown in a previous article by Overholdt in a recent issue of Normat. So let us leave this aside and consider the case of cubes.

A similar argument gives that the probability of being a cube should be $\frac{1}{3N^{2/3}}$. This leads to the integral

$$\frac{1}{3}\int_0^{(N/2)^{1/3}} \frac{dx}{(N-x^3)^{2/3}}$$

Making the substitution $y = \frac{x}{N^{1/3}}$ we reduce it to the integral

$$\frac{1}{3N^{1/3}}\int_0^{1/2^{1/3}} \frac{dy}{(1-y^3)^{2/3}}$$

This integral cannot be evaluated in closed form, but we can get an approximation of about 0.88. Now this sum involving the factor $\frac{1}{N^{1/3}}$ is asymptotically so small that we consider it safe to ignore the higher order terms. The probability of being the sum of two cubes for numbers the size of N is thus given by $CN^{-1/3}$ where we have computed the constant C to around 0.3. If we want a formula that gives the total number of sum of two cubes up to a limit N as say $\phi(N)$ we should have for a small number h compared to N that $\phi(N+h) - \phi(N) = CN^{-1/3}h$ because the right hand side counts the numbers in the interval $[N, N+h]$. Clearly it means that $\phi'(N) = CN^{-1/3}$, thus that $\phi(N) = C\frac{3}{2}N^{2/3} \sim 0.44N^{2/3}$.

This is clearly rather heuristic reasoning. How well does it stand up to reality? Let us compute this function up to $N = 10^6$ and compare with our predictions. The two curves are seen below plotted on a log/log scale. We seen that the rate of growth has about the right slope and also with the right multiplicative constant.

In particular our formula predicts about 4400 such numbers up to a million, the actual value is 4454

There is also a more direct and elementary way of estimating the frequency of sums of cubes. Consider a square with side $N^{1/3}$ consisting of numbers (x, y).

Clearly any number less than N involves cubes less than N. On the other side, not every pair in the square corresponds to a sum less than N, in fact the permissible pairs need to lie below a certain graph, in fact $y \leq (N - x^3)^{1/3}$. The number of such pairs is approximated by the integral

$$\int_0^{N^{1/3}} (N - x^3)^{1/3} dx$$

an obvious change of variables gives us

$$N^{2/3} \int_0^1 (1 - y^3)^{1/3} dy$$

Now we only need to approximate this integral and divide by two, as we are counting each number twice. The result is $0.439N^{2/3}$ which is very close to the one above, and far easier to motivate and make rigorous.

Now what is the probability that one number should be the sum of cubes in two different ways? Clearly if the probability of being a sum of two cubes at size N is given by $CN^{-1/3}$, being in two different ways should be $C^2N^{-2/3}$ which corresponds to a counting function $3C^2N^{1/3}$ which in our case would correspond to $0.3N^{1/3}$ which for $N = 10^6$ would give about 30 coincidences, in reality there are about 45. Now a million might be too low a number to give a fair reflection of a trend, but a more extensive count of cases up to a billion shows an even greater discrepancy. Instead of a predicted 300 of coincidences one documents 1570 cases. Could it be that being already the sum of two cubes makes it more likely to be the sum of two cubes in another way more likely? And if so why?

Now it may come as something of a surprise that one can indeed explicitly solve the diophantine equation, in the sense of making a complete parametrization of all solutions. However, this turns out to be less illuminating and useful, than one might at first suspect. It all hinges on the geometry of cubic surfaces.

Now the geometry of the cubic surface is classical, and has been the subject of classical treatieses as well as modern elementary presentations, and usually is presented as a salivating example in elementary texts on algebraic geometry and hence there is no need to delve into this in detail in this note, it will suffice to note the minimum of what is needed. For this purpose we start with a short digression on the geometric construction.

Skew-lines on a Cubic surface

The key-point is to find two skew lines. The existence of lines on a cubic surface can be shown by abstract formal reasoning (essentially a 'Fubini'type' argument) to actually find explicit ones on a given cubic is quite another matter. However, once one line is found, it is relatively easy to find the others. In our situation, the form of the cubic will be very simple, and this will not be a problem.

Now if given two skewlines L_1, L_2 , the lines that intersect them can be parametrized by $L_1 \times L_2$ which is isomorphic to $P^1 \times P^1$. This is immediate. Given any two points $P \in L_1, Q \in L_2$ they define a unique line L_{PQ}. Now this line will either be contained in the cubic, or residually intersect it in one point. One can show that there will be five such lines that are contained in the cubic, provided that we are willing to look at complex solutions. Conversely given a point outside two skew lines, there is a unique line through the point intersecting the two skew-lines. Namely consider the intersection of the two planes formed by the point and either of the two lines. Or if you prefer, projecting from the given point onto a plane, the projection of the two skew lines will intersect. Join the point with that intersection point. Disregarding those exceptional lines lying in the cubic, we will have a 1-1 correspondence between points P, Q (paramatrized by $P^1 \times P^1$ and points on the cubic. Technically one says that there is a birational map between the two varieties. More explicitly one can easily show that this parametrization can be given by four cubic forms in three variables, thus defining cubics on the projective plane, passing through six so called base points.

The special cubic $X^3 + Y^3 = Z^3 + W^3$

Integral solutions to the equation $X^3 + Y^3 = Z^3 + W^3$ are what we are looking for. Rational solutions are as good, as they lead to integral solutions by clearing denominators, because if (x, y, z, w) is a solution so is (tx, ty, tz, tw). This is just an illustration of the fact that the equation is homogenous. Geometrically one considers all of those equivalent, and speaks about the projective space P^3 consisting of homogeneous four-tuplets (excluding $(0, 0, 0, 0)$) and with the above equivalence. Thus among all rational solutions there is one (up to sign) canonical one, which is a primitive integral solution. That is a solution in integers which have no common factor.

The form of the cubic is very simple. It is said to be of Fermat type. It is very easy to write down all of the twenty-seven lines on it.

Namely considering

$$(X + Y)(X + \rho Y)(X + \rho^2 Y) = (Z + W)(Z + \rho W)(Z + \rho^2 W)$$

we get immediately $3 \times 3 = 9$ possibilities

$$(X + \rho^? Y) = (Z + \rho^k W) = 0$$

using the other two factorizations

$$(X - Z)(X - \rho Z)(X - \rho^2 Z) = (W - Y)(W - \rho Y)(W - \rho^2 Y)$$

and

$$(X - W)(X - \rho W)(X - \rho^2 W) = (Z - Y)(Z - \rho Y)(Z - \rho^2 Y)$$

we get the other 18

Two typical such lines are

$$\begin{array}{ll} X + \rho Y = 0 & Z + \rho W = 0 \\ X + \rho^2 Y = 0 & Z + \rho^2 W = 0 \end{array}$$

where ρ is a primitive cube root of one.

Those lines are skew, because if you want to find a solution to the four equations, you end up with the trivial one $(0,0,0,0)$ which is excluded. Furthermore they are not defined over the rational numbers $\mathbb{Q}$ but over the quadratic extension $\mathbb{Q}(\rho)$. Recall that ρ satisfies the quadratic equation $x^2+x+1=0$, i.e. we have the identity $\rho^2+\rho+1=0$. Now $\mathbb{Q}(\rho)$ comes with an involution, or if you prefer a conjugation, given by $\rho \to \rho^2$. This is an automorphism of fields and induced by complex conjugation. In this sense the two lines are skew. The conjugates of the points of one line make up the other line.

We will now consider points P on one line and their conjugates $\bar{P}$ on the other. The line L joining the two points will be defined over the reals, as its conjugate will intersect it in two points and hence coincide. In particular if $P \in \mathbb{Q}(\rho)$ the line will be defined over $\mathbb{Q}$. The cubic form above, defined over $\mathbb{Q}$ will restrict to a cubic binary form on L. It will have two conjugate zeroes $P, \bar{P}$, and the third residual zero (the residual intersection point) will hence be closed under conjugation and hence belong to $\mathbb{Q}$. Conversely if we have a rational point on our cubic, the unique line that goes through it, will intersect the two skew lines in conjugate points.

Now we can do this explicitly. The two lines can be parametrized respectively by $(u, -\rho^2 u, t, -\rho^2 t)$ and $(v, -\rho v, s, -\rho s)$, where $u, v, s, t \in \mathbb{Z}(\rho)$.

Now we parametrize the line L and it will be given by

$$\lambda(u, -\rho^2 u, t, -\rho^2 t) + \mu(v, -\rho v, s, -\rho s))$$

where $\lambda, \mu \in \mathbb{Z}(\rho)$

Plugging this into our cubic we get the binary cubic in λ, μ given by

$$(\lambda u + \mu v)^3 - (\lambda \rho^2 u + \mu \rho v)^3 - (\lambda t + \mu s)^3 + (\lambda \rho^2 t + \mu \rho s)^3$$

Simplifying this we note that (by design) the coefficients for λ^3 and μ^3 vanish and we are left with

$$\lambda\mu(\lambda(1-\rho^2)(u^2 v - t^2 s) + \mu(1-\rho)(uv^2 - ts^2))$$

We note that we have two bi-homogenous forms in $(u, t; v, s)$ given respectively by $(u^2 v - t^2 s)$ and $(uv^2 - ts^2)$ respectively. Bihomogenity is a generalization of bilinearity, fixing one set of homogenous co-ordinates it will be homogenous in the other. In fact they will be of bidgree $(2,1)$ and $(1,2)$ respectively. (Bilinear forms have bi-degree $(1,1)$). Two such forms will have five intersection points, meaning that they have common zeroes. For such values the cubic form becomes identicaly equal to zero, which means that the corresponding lines lie entirely on a cubic. In fact every two skew-lines on a cubic meet five other lines (necessarily skew). Two intersection points are obvious, namely $(0,1;1,0)$ and $(1,0;0,1)$ which can be denoted by $(\infty, 0)$ and $(0, \infty)$ respectively. If $uv \neq 0$ we can normalize them to 1. We then see immediately that $s = t$ and $s^3 = t^3 = 1$ which gives the three other lines corresponding to $(1,1), (\rho, \rho)$ and (ρ^2, ρ^2) respectively. Thus we see that three of the lines are real, and the two other complex conjugate. The case of $t = 1$ will correspond to the line L and give a 1-parameter family of trivial solutions given by $(\lambda+\mu, -\rho(\lambda\rho+\mu), \lambda+\mu, -\rho(\lambda\rho+\mu)$ from which you conclude that this is the line$X=Z, Y=W$

Disregarding those five cases, the binary cubic has three solutions, two complex conjugate and a third, which has to be invariant under conjugation, and hence be rational.

We get it by setting

$$\begin{aligned}\lambda &= (1-\rho)(uv^2-ts^2)\\ \mu &= (\rho^2-1)(u^2v-t^2s)\end{aligned}$$

So let us denote by $A(u,t;v,s)$ the bihomogenous form (uv^2-ts^2) and by B the one by switching the variables $B(u,t;v,s)=A(v,s;u,t)$. Furthermore as λ,μ are only defined up to a scalar multiple, it will be convenient to divide them by $\rho-\rho^2=\sqrt{-3}$ so we can write them

$$\begin{aligned}\lambda &= \rho^2(uv^2-ts^2)\\ \mu &= \rho(u^2v-t^2s)\end{aligned}$$

We can then get the parametrization of the cubic by

$$\begin{aligned}x &= \rho^2uA+\rho vB\\ y &= -(\rho uA+\rho^2vB)\\ z &= \rho^2tA+\rho sB\\ w &= -(\rho tA+\rho^2sB)\end{aligned}$$

Those will be four forms of bidegree $(2,2)$ spanning the linear space of all such form passing through the five intersection points $A=B=0$. Such a linear subspace is referred to as a linear system, and the points through which they all pass, the base points of the system. It is easy to find a basis of 9 forms of bidegree $(2,2)$ and each base point imposes one linear condition. They are all independent and the system will have dimension four, and we have just exhibited an explicit base.

Those four forms give a map from $P^1\times P^1$ to P^4 defined outside the five-base points, with the image the cubic surface. One says that the five base-points are blown up to five lines on the cubic.

But now we are interested in the rational solutions, i.e. of $\mathbb{Q}$ not just $\mathbb{Q}(\rho)$. A sufficient condition for that is that $\overline{A}=B$ which means that $s=\bar{t}$ and $v=\bar{u}$

We then get

$$\begin{aligned}x &= -|u|^2+<u,t>|t|\\ y &= |u|^2-<t,u>|t|\\ z &= |t|^2-<t,u>|u|\\ w &= -|t|^2+<u,t>|u|\end{aligned}$$

where $<u,t>=-(\rho^2u\bar{t}+\rho\bar{u}t)=-2Re(\rho^2u\bar{t})$ which is a bilinear form over $\mathbb{Q}$ and satisfies $<su,t>=<u,\bar{s}t>$ from which follows that $<su,st>=|s|<u,t>$ and in particular $<u,u>=|u|$. We note that $<u,t>=<t,u>$ iff $u\bar{t}$ is real or equivalently t/u is real. In that case we have the trivial solution $x+y=0$ and $z+w=0$.

We can now systematically plug in values t/u for elements $t,u\in\mathbb{Z}[\rho]$ of simple numerators and denominators, meaning that we bound their norms. We talk about fractions of bounded heights. By multiplying with a unit in the ring we can assume that t lies in the sector spanned by 1 and $1+\rho$. We will only consider examples when t, u are relatively prime.

We get the following list

$15^3 + 33^3 = 34^3 + 2^3$	39312	$(2)/(-1-2\rho)$
$15^3 + 9^3 = 16^3 + 2^3$	4104	$(2)/(1-\rho)$
$24^3 + 165^3 = 157^3 + 86^3$	4505949	$(2+3\rho)/(3)$
$3^3 + 60^3 = 22^3 + 59^3$	216027	$(2+3\rho)/(3+3\rho)$
$59^3 + 184^3 = 186^3 + 3^3$	6434883	$(3)/(-2-3\rho)$
$20^3 + 304^3 = 276^3 + 192^3$	28102464	$(3)/(-1-4\rho)$
$20^3 + 223^3 = 159^3 + 192^3$	11097567	$(3)/(1-3\rho)$
$86^3 + 76^3 = 102^3 + 24^3$	1075032	$(3)/(1-2\rho)$
$90^3 + 456^3 = 457^3 + 47^3$	95547816	$(3+4\rho)/(2-2\rho)$
$108^3 + 516^3 = 489^3 + 279^3$	138647808	$(3+4\rho)/(4)$
$168^3 + 222^3 = 241^3 + 119^3$	15682680	$(3+4\rho)/(4+2\rho)$
$4^3 + 152^3 = 41^3 + 151^3$	3511872	$(3+4\rho)/(4+4\rho)$
$186^3 + 1125^3 = 1117^3 + 332^3$	1430262981	$(3+5\rho)/(4-\rho)$
$352^3 + 788^3 = 809^3 + 151^3$	532918080	$(3+5\rho)/(4)$
$420^3 + 549^3 = 621^3 + 42^3$	239557149	$(3+5\rho)/(4+\rho)$
$382^3 + 245^3 = 413^3 + 16^3$	70449093	$(3+5\rho)/(4+3\rho)$
$276^3 + 180^3 = 297^3 + 87^3$	26856576	$(3+5\rho)/(4+4\rho)$
$87^3 + 873^3 = 864^3 + 276^3$	665997120	$(4)/(-3-5\rho)$
$151^3 + 617^3 = 620^3 + 4^3$	238328064	$(4)/(-3-4\rho)$
$135^3 + 825^3 = 760^3 + 500^3$	563976000	$(4)/(-1-5\rho)$
$15^3 + 945^3 = 756^3 + 744^3$	843912000	$(4)/(-5\rho)$
$135^3 + 633^3 = 508^3 + 500^3$	256096512	$(4)/(1-4\rho)$
$279^3 + 297^3 = 360^3 + 108^3$	47915712	$(4)/(1-3\rho)$
$151^3 + 425^3 = 332^3 + 352^3$	80208576	$(4)/(2-3\rho)$
$42^3 + 504^3 = 378^3 + 420^3$	128098152	$(4+\rho)/(2-3\rho)$
$119^3 + 385^3 = 378^3 + 168^3$	58751784	$(4+2\rho)/(1-3\rho)$
$47^3 + 97^3 = 66^3 + 90^3$	1016496	$(4+2\rho)/(4+\rho)$
$16^3 + 686^3 = 644^3 + 382^3$	322832952	$(4+3\rho)/(2-3\rho)$
$206^3 + 1180^3 = 1182^3 + 72^3$	1651773816	$(4+5\rho)/(2-3\rho)$
$279^3 + 2241^3 = 2242^3 + 188^3$	11276201160	$(4+6\rho)/(3-3\rho)$

We note that the Ramanujam example does not occur, in fact it will never occur, because the parametrization only works up to a multiple. Even if the numerator and denominator of the fraction is reduced, that does not have to be the case with the image. In fact corresponding to